Basic Review of Growth Models

- Deterministic and finite horizon growth model
  - Single agent
  - Decision: how much to consume and save in each period
  - Time: $t = 0, 1, \dots, T$

  - Production function

$$\underbrace{Y_t}_{\text{output}} = F\left(\underbrace{K_t}_{\text{capital}}, \underbrace{N_t}_{\text{labor}}\right)$$

    - Output can be used for either consumption or saving
    - Properties of F:
      1) $F(0,0) = 0 \rightarrow$ no free lunch
         - Stronger version: $F(0,\cdot) = 0$ or $F(\cdot\,,0) = 0$
      2) Strictly increasing in both arguments
      3) Strictly concave in both arguments
         - To be consistent with data, should also assume homogeneity of degree 1
      4) Twice continuously differentiable

  - Resource constraint:

$$\underbrace{C_t}_{\text{consumption}} + K_{t+1} \leq F(K_t, N_t) + \left(1 - \underbrace{\delta}_{\text{depreciation rate}}\right)K_t$$

  - Objective: maximize $U(c_0, c_1, \dots, c_T)$
    - Note that utility does not have to be time separable.
    - E.g.

$$U(c_0, c_1, \dots, c_T) = \sum_{t=0}^{T} \beta^t u(c_t)$$

$$U(c_0, c_1, \dots, c_T) = \left[\sum_{t=0}^{T} c_t^{\rho}\right]^{1/\rho}$$

    - Note that leisure does not enter the objective function
      $\Longrightarrow$ agent works his whole time endowment : $N_t = N$
      Let $f(K_t) = F(K_t, N) + (1-\delta)K_t \rightarrow$ assume $f(0) = 0$ (the stronger version)

  - Problem of the Agent

$$\max_{\{c_0, c_1, \dots, c_T; K_1, \dots, K_T\}} U(c_0, \dots, c_T)$$

  Subject to

$$\begin{aligned} C_t + K_{t+1} \leq f(K_t) \quad & t = 0, \dots, T \\ C_t \geq 0 \quad & t = 0, \dots, T \\ K_{t+1} \geq 0 \quad & t = 0, \dots, T \\ K_0 > 0 \ \text{ given} & \end{aligned}$$

Deterministic, Finite Horizon Growth Model

- Problem of the agent

$$\max_{\{C_t,K_{t+1}\}_{t=0}^T} U(C_0,\dots,C_T)$$

Subject to

$$\begin{aligned} f(K_t) - C_t - K_{t+1} &\geq 0 \quad t = 0,\dots,T \\ C_t &\geq 0 \quad t = 0,\dots,T \\ K_{t+1} &\geq 0 \quad t = 0,\dots,T \end{aligned}$$

$$K_0 > 0 \text{ given}$$

$$f(K_t) = F(K_t, N) + (1-\delta)K_t, \qquad \delta \in [0,1]$$

$$f(0) = 0$$

- To solve this problem, use a Lagrangean:

$$\max_{\{C_t,K_{t+1},\lambda_t,\mu_t,\omega_t\}_{t=0}^T} U(C_0,\dots,C_T) + \sum_{t=0}^{T}\{\lambda_t(f(K_t) - K_{t+1} - C_t) + \mu_t C_t + \omega_t K_{t+1}\}$$

with $K_0 > 0$ given.

First order conditions:

$$C_t: \qquad \frac{\partial U(C_0,\dots,C_T)}{\partial C_t} - \lambda_t + \mu_t = 0 \qquad \forall t = 0,\dots,T$$

$$K_{t+1}: \quad -\lambda_t + \lambda_{t+1}\frac{\partial f(K_{t+1})}{\partial K_{t+1}} + \omega_t = 0 \quad \forall t = 0,\dots,T-1$$

$$K_{T+1}: \quad -\lambda_T + \omega_T = 0 \qquad t = T+1$$

- λ_t shows how tight the resource constraint is binding.
- μ_t appears in consumption FOC so that it holds with strict equality.
- At time $T+1$, the constraint

$$f(K_{T+1}) - K_{T+2} - C_{T+1} \geq 0$$

does not exist. Consider the Lagrangian:

$$\begin{aligned} U(C_0,\dots,C_T) &+ \{\lambda_0(f(K_0) - K_1 - C_0) + \mu_0 C_0 + \omega_0 K_1\} + \cdots \\ &+ \{\lambda_{T-1}(f(K_{T-1}) - K_T - C_{T-1}) + \mu_{T-1}C_{T-1} + \omega_{T-1}K_T\} \\ &+ \underbrace{\{\lambda_T(f(K_T) - K_{T+1} - C_T) + \mu_T C_T + \omega_T K_{T+1}\}}_{\text{constraint at the terminal period}} \end{aligned}$$

So when taking derivative w.r.t. to K_{T+1}, we only get

$$-\lambda_T + \omega_T = 0!!!!$$

Complementary slackness conditions:

$$\begin{aligned} \lambda_t[f(K_t) - K_{t+1} - C_t] = 0, &\qquad \lambda_t \geq 0, \quad \forall t = 0,\dots,T \\ \mu_t C_t = 0, &\qquad \mu_t \geq 0, \quad \forall t = 0,\dots,T \\ \omega_t K_{t+1} = 0, &\qquad \omega_t \geq 0, \quad \forall t = 0,\dots,T \end{aligned}$$

A sufficient condition for consumption to be strictly positive for all time periods:

$$\lim_{C_t \to 0} \frac{\partial U(C_0,\dots,C_T)}{\partial C_t} = \infty, \qquad \forall t = 0,\dots,T$$

$$\Rightarrow \ C_t > 0, \qquad \forall t = 0,\dots,T$$

$$\Rightarrow \ \mu_t = 0, \qquad \forall t = 0,\dots,T$$

- This is the first part of the ***Inada Condition*** (the other part is to guarantee that consumption is bounded).

Since $\mu_t = 0$,

$$\lambda_t = \frac{\partial U(C_0, \dots, C_T)}{\partial C_t} > 0$$

thus the resource constraint binds in all periods (i.e. the agent never lets any resources go to waste).

Given $C_t > 0$ in all t, and $f(0) = 0$, it follows that $K_{t+1} > 0$ for all $t = 0, \dots, T-1$. This implies that

$$\omega_t = 0, \qquad \forall t = 0, \dots, T-1$$

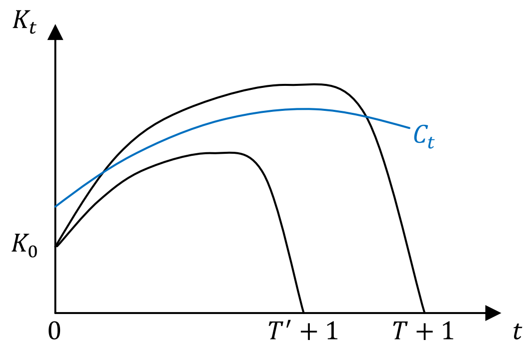
At T, $\lambda_t = \omega_t$. Since $\lambda_T > 0$, it follows that $\omega_T > 0$, and hence $K_{T+1} = 0$ (from complementary slackness condition).

Substitute λ_t into the K_{t+1} FOC:

$$\lambda_t = \frac{\partial U(C_0, \dots, C_T)}{\partial C_t}$$

$$\Rightarrow \begin{cases} -\dfrac{\partial U(C_0, \dots, C_T)}{\partial C_t} + \dfrac{\partial U(C_0, \dots, C_T)}{\partial C_{t+1}} \cdot \dfrac{\partial f(K_{t+1})}{\partial K_{t+1}} = 0 & t = 0, \dots, T-1 \\ f(K_t) - K_{t+1} - C_t = 0 & t = 0, \dots, T \\ K_{T+1} = 0 & t = T \end{cases}$$

Together with $K_0 > 0$ given, can calculate $\{C_t, K_{t+1}\}_{t=0}^{T}$.



- ❖ Moving to the infinite horizon, we need the utility function to have some recursive structure:

$$U_t = \underbrace{u(c_t)}_{\text{period utility}} + \underbrace{\beta}_{\text{constant factor}} U_{t+1}, \qquad \beta \in (0,1)$$

with $u(C_t) : [0, \infty) \to \mathbb{R}$ being

- Strictly increasing
- Strictly concave
- Twice continuously differentiable

➢ Example. Let

$$U_t = \sum_{s=0}^{\infty} \beta^s u(C_{t+s})$$

- It is sufficient to assume that C_t is bounded, and from the concavity of $u(\cdot)$, it follows that $u(C_t)$ is also bounded.

Problem of the Agent in Infinite Horizon

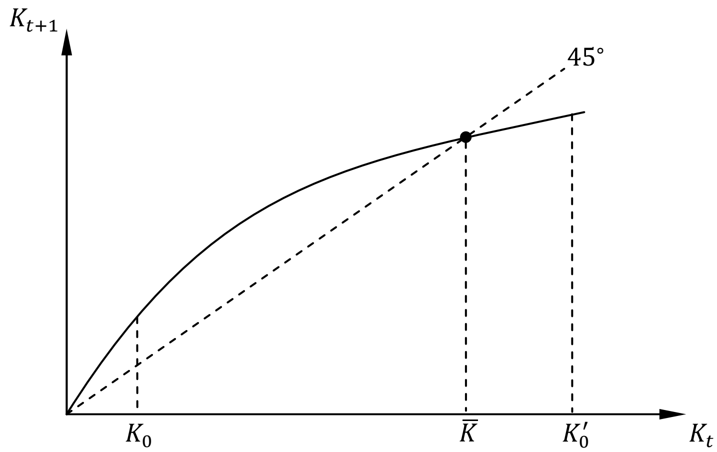
- Objective:

$$\max_{\{C_t,K_{t+1}\}_{t=0}^{\infty}} U_0 = \sum_{t=0}^{\infty} \beta^t u(C_t)$$

Subject to

$$f(K_t) - K_{t+1} - C_t \geq 0 \quad t = 0, \dots, \infty$$
$$C_t \geq 0 \quad t = 0, \dots, \infty$$
$$K_{t+1} \geq 0 \quad t = 0, \dots, \infty$$
$$K_0 \text{ given}$$

- If there is no discount factor β, the infinite sum diverges to infinity (in most cases given the assumption of $u(\cdot)$), then there is no way to compare different consumption streams.
- The assumption we have made guarantee that $u(C_t)$ is bounded. This is because C_t is bounded, which follows from the fact that K_t is bounded:
  - Assume that $f(K_t) = K_t^{\alpha}$. Then resource constraint is $K_t^{\alpha} = K_{t+1} + C_t$. Suppose $C_t = 0$. Then, $K_t^{\alpha} = K_{t+1}$.



If $K_0 < \bar{K}$, then $K_t < K_{t+1}$ and $K_t \to \bar{K}$

If $K_0 > \bar{K}$, then $K_t > K_{t+1}$ and $K_t \to \bar{K}$, where $\bar{K}^{\alpha} = \bar{K} \;\Rightarrow\; \bar{K} = 1$

Thus, $K^{MAX} = \max\{\bar{K}, K_0\}$ and

$$K_t < \infty \;\Rightarrow\; C_t < \infty \;\Rightarrow\; U_0 < \infty$$

for any feasible $\{C_t, K_{t+1}\}_{t=0}^{\infty}$ given $K_0 > 0$.

  - From this it is apparent that diminishing marginal product is crucial.

- Reformulate the problem in terms of Lagrangean:

$$\max_{\{C_t,K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \{\beta^t u(C_t) + \lambda_t[f(K_t) - K_{t+1} - C_t] + \mu_t C_t + \omega_t K_{t+1}\}, \qquad K_0 \text{ given}$$

First-order conditions:

$$C_t: \quad \beta^t u_C(C_t) - \lambda_t + \mu_t = 0$$
$$K_{t+1}: \quad -\lambda_t + \lambda_{t+1} f_K(K_{t+1}) + \omega_t = 0$$

Complementary slackness conditions: for all $t = 0, \dots, \infty$,

$$\lambda_t[f(K_t) - K_{t+1} - C_t] = 0$$

$$\mu_t C_t = 0$$
$$\omega_t K_{t+1} = 0$$
$$\lambda_t \geq 0, \qquad \mu_t \geq 0, \qquad \omega_t \geq 0$$

- Note that these are only the necessary condition of the maximum, but they are not *sufficient* to guarantee a solution.

- If we have ***Inada conditions*** on $u(\cdot)$:
  1) $\lim_{C \to 0} u_C(C) = \infty$
  2) $\lim_{C \to \infty} u_C(C) = 0$

$$\Rightarrow \; C_t > 0 \;\Rightarrow\; \mu_t = 0$$
$$\Rightarrow \; \lambda_t = \beta^t u_C(C_t) > 0$$
$$\Rightarrow \; K_{t+1} > 0, \qquad \omega_t = 0$$
$$\Rightarrow \begin{cases} -\beta^t u_C(C_t) + \beta^{t+1} u_C(C_{t+1}) f_K(K_{t+1}) = 0 & t = 0, \dots, \infty \\ f(K_t) - K_{t+1} - C_t = 0 & t = 0, \dots, \infty \\ K_0 > 0 \text{ given} & \end{cases}$$

- Recall in finite horizon economy, the FOC w.r.t K_{T+1} is

$$-\lambda_T + \omega_T = 0 \;\Rightarrow\; \lambda_T = \omega_T > 0$$

  We also have

$$\lambda_T = \beta^T u_C(C_T)$$
$$\omega_T K_{T+1} = 0$$

  All three conditions imply that

$$\lambda_T K_{T+1} = 0 \;\Rightarrow\; \beta^T u_C(C_T) K_{T+1} = 0$$

  Then, the transversality condition (TVC) is naturally derived:

$$\lim_{T \to \infty} \beta^T u_C(C_T) K_{T+1} = 0.$$

  This builds on the intuition that an infinite horizon model is the limit of the finite horizon model.
  - Identify the conditions that are particular in the last period of the finite horizon economy, take them to the limit, and we will get the sufficient conditions that guarantee the solution (cf. Stockey & Lucas Thm 4.15).

Recursive Formulation of the Growth Model

- Restating the problem

$$\max_{\{C_t,K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t), \qquad s.t. \begin{cases} f(K_t) - C_t - K_{t+1} \geq 0 & t = 0, \dots, \infty \\ C_t \geq 0 & t = 0, \dots, \infty \\ K_{t+1} \geq 0 & t = 0, \dots, \infty \\ K_0 \text{ given} \end{cases}$$

- Recursive Formulation:
  - ***State variable***: K
    - Notice that we are dropping the time subscript in the recursive formulation
  - ***Control variables***: C, K'
    - We use *prime* ($'$) to denote the next period's variable.
  - To reformulate the problem:

$$V(K) = \max_{C,K'} u(C) + \beta V(K'), \qquad s.t. \begin{cases} f(K) - K' - C \geq 0 \\ C \geq 0 \\ K' \geq 0 \end{cases}$$

    - Notice that K_0 given is not one of the constraints. The recursive formulation holds for any given level of capital.
    - Since the constraints hold with equality, it follows that

$$V(K) = \max_{0 \leq K' \leq f(K)} u(f(K) - K') + \beta V(K')$$

    - Given u and f are twice differentiable, strictly increasing, strictly concave, and that there exists K^{MAX} (as defined in the previous lecture), the following results hold:
      1) The V function exists, is differentiable, strictly increasing, and strictly concave.
      2) $K' = g(K)$, where g is time-invariant, increasing, and differentiable.
      3) $V(K)$ is the limit to (as $s \to \infty$)

$$V^{s+1}(K) = \max_{0 \leq K' \leq f(K)} u(f(K) - K') + \beta V^s(K') \quad \text{with } V^0(K) = 0$$

- Since the non-negativity constraint is not going to bind, the problem becomes

$$V(K) = \max_{K'} u(f(K) - K') + \beta V(K')$$

FOC:

$$K': \quad \underbrace{-u_c(f(K) - K')}_{\substack{\text{cost in terms} \\ \text{of today's utility}}} + \underbrace{\beta V_K(K')}_{\substack{\text{PV of the} \\ \text{benefit tomorrow}}} = 0$$

Recall that $K' = g(K) = \arg\max_{K'} u(f(K) - K') + \beta V(K')$:

$$V(K) \equiv u\big(f(K) - g(K)\big) + \beta V\big(g(K)\big)$$

Taking derivative of V w.r.t. K yields

$$\begin{aligned} V_K(K) &= u_C\big(f(K) - g(K)\big)[f_K(K) - g_K(K)] + \beta V_K\big(g(K)\big) g_K(K) \\ &= u_C\big(f(K) - g(K)\big) f_K(K) + \underbrace{\big[-u_C\big(f(K) - g(K)\big) + \beta V_K\big(g(K)\big)\big]}_{=0 \text{ by the FOC of } K'} g_K(K) \\ &= u_C\big(f(K) - g(K)\big) f_K(K) > 0 \end{aligned}$$

  - Why does g_K not matter? Notice that g_K is the marginal effect of saving. V_K measures how a marginal change of capital stock influences utility. So when there is an increase in K, there is extra resources, while the marginal decisions over consumption and saving are already handled optimally by plugging in the g function.

Note that

$$V_K(K') = u_C\big(f(K') - g(K')\big)f_K(K')$$

Then the FOC becomes

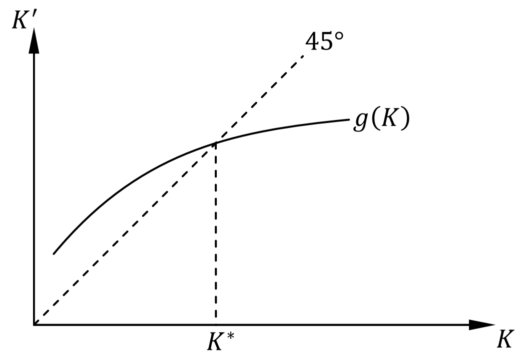
$$-u_C(f(K) - K') + \beta u_C\big(f(K') - g(K')\big)f_K(K') = 0$$

The optimal solution is $g(K)$ such that

$$-u_C\big(f(K) - g(K)\big) + \beta u_C\Big(f\big(g(K)\big) - g\big(g(K)\big)\Big)f_K\big(g(K)\big) = 0, \qquad \forall K$$

Alternatively, the optimal solution is $C(K)$ and $g(K)$ such that

$$\begin{cases} f(K) - C(K) - g(K) = 0 \\ -u_C\big(C(K)\big) + \beta u_C\Big(C\big(g(K)\big)\Big)f_K\big(g(K)\big) = 0 \end{cases} \qquad \forall K$$



At $K^* = g(K^*)$:

$$-u_C(f(K^*) - K^*) + \beta u_C(f(K^*) - K^*)f_K(K^*) = 0 \;\Rightarrow\; 1 - \beta f_K(K^*) = 0$$

- Uniqueness of a solution to this problem comes from the fact that f is strictly concave (i.e. f_K is strictly decreasing).
- Existence of the solution is guaranteed if f satisfies Inada conditions.

Thus, a unique steady state exists. When we define a domain, we have to make sure that it encompasses the steady state value.

Stochastic Growth Model

- Assume $Y_t = Z_t F(K_t, \bar{N})$, where Z_t is subject to a stochastic process
  - The Resource Constraint:

    $$C_t + K_{t+1} = Z_t F(K_t, \bar{N}) + (1-\delta)K_t$$

    Z_t is a random variable, known at the beginning of period t (i.e. Z_{t+1} is not known in t)

  - $\{Z_t\}_{t=0}^{\infty}$ is a stochastic process (assume further that this is a Markov process), and can be either continuous or discrete
    1) Continuous: $Z_t \in \left(\underline{Z}, \bar{Z}\right)$
    2) Discrete: $Z_t \in \{Z^1, Z^2, \dots, Z^n\}$

  - Markov assumption: to form expectation over Z_{t+1}, we only need to know Z_t

    $$V(K,Z) = \max_{C,K'} u(C) + \beta E[V(K',Z'|Z)], \qquad s.t. \begin{cases} C + K' = ZF(K,\bar{N}) + (1-\delta)K \\ Z \sim \underbrace{Q(Z,Z')}_{\text{transition function}} \end{cases}$$

    - Solution: $g(K,Z)$

Stochastic Growth Model (cont'd)

- Problem of the agent (in recursive form):

$$V(K,z)=\max_{C,K'} u(C)+\beta E[V(K',z')|z], \qquad s.t. \begin{cases} C+K'=zF(K,\bar{N})+(1+\delta)K \\ z'\sim \underbrace{Q(z,z')}_{\text{transition function}} \end{cases}$$

  - Recall that $Output = z_t F(K_t,\bar{N})$, where z_t is the ***total factor productivity*** (***TFP***)
  - $\{z_t\}_{t=0}^{\infty}$ follows a stochastic Markov process

    1) Discrete case (Markov chain)

    $$Z=\{z^1,\dots,z^n\}, \qquad \text{where } n\in\mathbb{N}$$

    Z is the set of all possible realization of z. Π is an $n\times n$ transition matrix with element π_{ij}

    $$\pi_{ij}=\text{Prob}\left(z'=z^j\middle|z=z^i\right)$$

    Now the problem becomes

    $$V\left(K,z^i\right)=\max_{C,K'} u(C)+\beta\sum_{j=1}^{n}\pi_{ij}V\left(K',z^j\right),$$
    $$s.t.\ C+K'=z^iF(K,\bar{N})+(1-\delta)K, \qquad \forall K,\forall i=1,\dots,n$$

    - Example. Suppose

    $$Z=\{z^L,z^H\}, \qquad \Pi=\begin{bmatrix} \pi_{LL} & \overbrace{1-\pi_{LL}}^{\pi_{LH}} \\ \underbrace{1-\pi_{HH}}_{\pi_{HL}} & \pi_{HH} \end{bmatrix}$$

    $$\underbrace{V^L(K)}_{=V(K,z^L)}=\max_{C^L,K^L} u(C^L)+\beta\{\pi_{LL}V^L(K^L)+(1-\pi_{LL})V^H(K^L)\},$$
    $$s.t.\ C^L+K^L=z^LF(K,\bar{N})+(1-\delta)K, \qquad \forall K$$
    $$V^H(K)=\max_{C^H,K^H} u(C^H)+\beta\{(1-\pi_{HH})V^L(K^H)+\pi_{HH}V^H(K^H)\},$$
    $$s.t.\ C^H+K^H=z^HF(K,\bar{N})+(1-\delta)K, \qquad \forall K$$

    2) Continuous case

    $$Z=[\underline{z},\bar{z}], \qquad 0<\underline{z}<\bar{z}<\infty$$

    with the associated σ-algebra being $\mathcal{Z}$. The transition function is defined by

    $$Q:Z\times\mathcal{Z}\to[0,1]$$

    For example, $Q(a,A)=\text{Prob}(z'=A|z=a)$. Then the problem becomes

    $$V(K,z)=\max_{C,K'} u(C)+\beta\int_{\underline{z}}^{\bar{z}}V(K',z')Q(z,dz'),$$
    $$s.t.\ C+K'=z'F(K,\bar{N})+(1-\delta)K$$

Endogenous Labor/Leisure

- Let L_t denote leisure, and $\bar{L}$ be the time available for work and leisure in a period (i.e. endowment of time).

$$L_t + N_t = \bar{L}$$

  - Note that the time constraint holds with *equality* here!
  - The time constraint can be more complicated by including other uses of time, e.g. non-market labor.
  - $\bar{L}$ is sometimes referred to as "discretionary time"
  - Frequently, $\bar{L} = 1$

- Utility function: $u(C_t, L_t)$.
  - Assumptions:
    (1) Twice continuously differentiable
    (2) Strictly increasing
    (3) Strictly concave in both arguments
    (4) Inada conditions

- Problem of the agent (deterministic case):

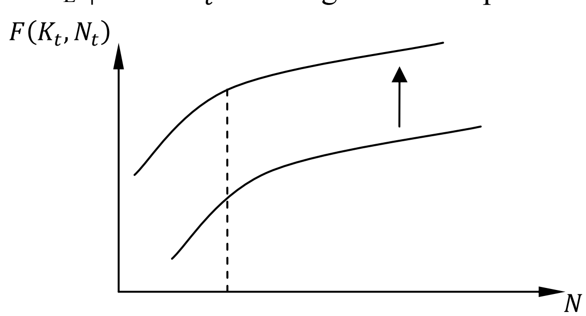
$$\max_{\{C_t, K_{t+1}, L_t, N_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t, L_t),$$

$$s.t. \begin{cases} C_t + K_{t+1} \leq F(K_t, N_t) + (1-\delta)K_t & t \in \mathbb{N} \cup \{0\} \\ L_t + N_t = \bar{L} & t \in \mathbb{N} \cup \{0\} \\ C_t \geq 0 & t \in \mathbb{N} \cup \{0\} \\ K_{t+1} \geq 0 & t \in \mathbb{N} \cup \{0\} \\ L_t \geq 0 & t \in \mathbb{N} \cup \{0\} \\ N_t \geq 0 & t \in \mathbb{N} \cup \{0\} \\ K_0 \text{ given} & \end{cases}$$

  - Given Inada conditions on u, it is sufficient to just assume $F(0,\cdot) = 0$ to guarantee interior solution for the choice variables.

- Suppose the marginal product of labor evolves ~~stochastically~~ over time. How does labor react to a temporary increase (or decrease) in the marginal productivity of labor?

$$\text{MP}_\text{L} \uparrow \;\Rightarrow\; \forall K_t \text{ we can get more output of } N_t$$



  - 3 effects:
    (1) Income effect: agents want more consumption and leisure $\rightarrow \downarrow N_t$

(2) Substitution effect: opportunity cost of leisure is higher $\rightarrow \uparrow N_t$
(3) Intertemporal substitution effect: $\uparrow N_t$ today $\rightarrow$ accumulate more capital $\rightarrow$ save on labor tomorrow
- Temporarily more productive
  → work more and save more capital for tomorrow
  → tomorrow can work less

❖ 3 margins (or trade-offs) in growth model
(1) C_t v.s. C_{t+1}
(2) C_t v.s. L_t
(3) L_t v.s. L_{t+1}

➢ C_t v.s. C_{t+1}

$$\max_{\{C_t,K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t), \qquad s.t. \begin{cases} C_t + K_{t+1} = f(K_t) \\ K_0 \text{ given} \end{cases}$$

FOC:

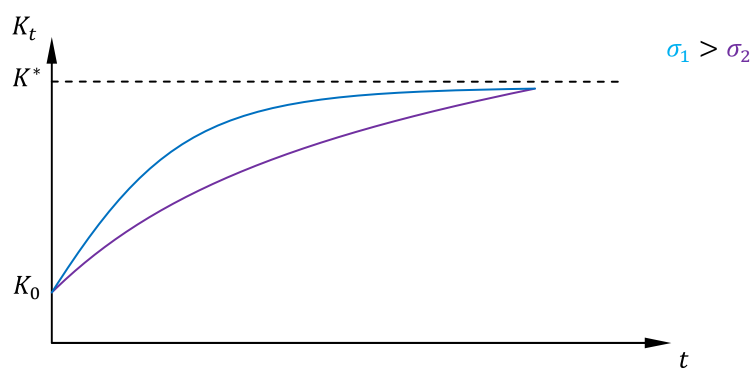
$$-u_C(C_t) + \beta u_C(C_{t+1}) f_K(K_{t+1}) = 0$$

Steady state:

$$-1 + \beta f_K(K^*) = 0$$

Since K^* is independent of u, does the Intertemporal trade-offs of C matter? Suppose

$$u(C) = \frac{C^{1-\sigma} - 1}{1-\sigma}, \qquad \sigma > 0$$



- To get to K^* faster, we need to sacrifice more consumption today. A higher σ implies that it is more painful to give up consumption today for consumption tomorrow.

➢ C_t v.s. L_t. Abstract from capital:

$$\max_{\{C_t,N_t,L_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t, L_t), \qquad s.t. \begin{cases} C_t = z_t N_t \\ N_t + L_t = 1 \end{cases}$$

$$\Rightarrow \max_{N_t} u(z_t N_t, 1 - N_t)$$

FOC

$$u_C(C_t) z_t - u_L(1 - N_t) = 0, \qquad \text{where } C_t = z_t N_t$$

How does labor depend on z_t? Suppose $\uparrow z_t$.
- Income effect: $\downarrow N_t$

- Substitution effect: $\uparrow N_t$

Suppose z_t has a trend.

- If income effect dominates: $N_t \to 0$
- If substitution effect dominates: $N_t \to 1$

In general, if the utility function is

$$u(C, L) = \frac{C^{1-\sigma}}{1-\sigma} v(L), \qquad \text{where } v(L) \text{ is strictly increasing and strictly concave}$$

Then, N_t does not depend on z_t (income and substitution effects cancel).

- L_t v.s. L_{t+1}.

$$\max_{\{C_t, K_{t+1}, N_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t, 1 - N_t), \qquad s.t. \begin{cases} C_t + K_{t+1} = z_t F(K_t, N_t) + (1-\delta) K_t \\ K_0 > 0 \text{ given} \end{cases}$$

We have all three effects:

- L_t v.s. L_{t+1}: Intertemporal substitution effect
  $\to$ if z_t is temporarily higher
  $\to \uparrow N_t$ to $\uparrow K_{t+1}$
  $\to \downarrow N_{t+1}$
  - Capital allows to save on labor in "bad" times
  - With capital, even if we assume a utility function that leads to the cancellation of the income and substitution effect, there is still the avenue that affects labor decisions through capital accumulation.

Economic Growth (cont'd)

❖ Labor augmenting production function:

$$Y_t = z_t F(K_t, \gamma^t N_t)$$

with labor-augmented productivity γ_t.

❖ Utility function ($\sigma > 0, \sigma \neq 1$):

$$u(C,L) = \frac{C^{1-\sigma}}{1-\sigma} v(L)$$

- Alternative ($\sigma = 1$),

$$u(C,L) = \ln C + v(L)$$

- C.f. King, Plossor, Rebelo (1988).

❖ Problem of the agent:

$$\max_{\{C,K_{t+1},N_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma} v(1-N_t), \qquad s.t. \begin{cases} C_t + K_{t+1} = F(K_t, \gamma^t N_t) + (1-\delta)K_t \\ K_0 > 0 \text{ given} \end{cases}$$

❖ Detrending. Let $\hat{X}_t = \frac{X_t}{\gamma^t}$.

- Detrending resource constraint:

$$\frac{C_t}{\gamma^t} + \frac{K_{t+1}}{\gamma^{t+1}}\gamma = \frac{1}{\gamma^t} F(K_t, \gamma^t N_t) + (1-\delta)\frac{K_t}{\gamma^t} \;\Rightarrow\; \hat{C}_t + \gamma \hat{K}_{t+1} = F(\hat{K}_t, N_t) + (1-\delta)\hat{K}_t$$

- Utility function:

$$\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma} v(1-N_t) \cdot \frac{\gamma^{t(1-\sigma)}}{\gamma^{t(1-\sigma)}} \;\Rightarrow\; \sum_{t=0}^{\infty} \beta^t \frac{\hat{C}_t^{1-\sigma}}{1-\sigma} v(1-N_t)\gamma^{t(1-\sigma)}$$

$$\Rightarrow \sum_{t=0}^{\infty} \underbrace{[\beta\gamma^{1-\sigma}]}_{\substack{\tilde{\beta} \equiv \beta\gamma^{1-\sigma} \\ \tilde{\beta} \in (0,1)}}{}^{t} \frac{\hat{C}_t^{1-\sigma}}{1-\sigma} v(1-N_t)$$

❖ The problem becomes

$$\max_{\{\hat{C},\hat{K}_{t+1},\hat{N}_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \tilde{\beta}^t \frac{\hat{C}_t^{1-\sigma}}{1-\sigma} v(1-N_t), \qquad s.t. \begin{cases} \hat{C}_t + \gamma \hat{K}_{t+1} = F(\hat{K}_t, \hat{N}_t) + (1-\delta)\hat{K}_t \\ \hat{K}_0 > 0 \text{ given} \end{cases}$$

- FOC's:

$$\hat{K}_{t+1}: \quad -\gamma\tilde{\beta}^t \hat{C}_t^{-\sigma} v(1-N_t) + \tilde{\beta}^{t+1}\hat{C}_{t+1}^{-\sigma}\left[F_k(\hat{K}_{t+1}, N_{t+1}) + 1 - \delta\right] v(1-N_{t+1}) = 0$$

$$N_t: \quad \tilde{\beta}^t \frac{\hat{C}_t^{1-\sigma}}{1-\sigma} v_L(1-N_t) + \tilde{\beta}^t\, \hat{C}_t^{-\sigma} v(1-N_t) F_n(\hat{K}_t, N_t) = 0$$

$$\hat{C}_t + \gamma \hat{K}_{t+1} = F(\hat{K}_t, N_t) + (1-\delta)\hat{K}_t$$

$$K_0 > 0 \text{ given}$$

$$TVC$$

We need to write the resource constraint because we used $\hat{C}_t$ in the FOC's, so we need the constraint to specify what $\hat{C}_t$ is. → small price for neatness.

- Steady state: $\hat{C}_t = \hat{C}^*$, $\widehat{K}_t = \widehat{K}^*$, $N_t = N^*$. Impose on FOC

$$-\gamma {\hat{C}^*}^{-\sigma} v(1-N^*) + \tilde{\beta} {\hat{C}^*}^{-\sigma} v(1-N^*)\left[F_k\left(\widehat{K}^*, N^*\right) + 1 - \delta\right] = 0$$

$$\Rightarrow\ -1 + \frac{\tilde{\beta}}{\gamma}\left[F_k\left(\widehat{K}^*, N^*\right) + 1 - \delta\right] = 0$$

$$\Rightarrow\ -1 + \frac{\beta}{\gamma^{\sigma}}\left[F_k\left(\widehat{K}^*, N^*\right) + 1 - \delta\right] = 0$$

  - Recall in the non-detrended model, the steady state is

$$-1 + \beta[F_k(K^*, N^*) + 1 - \delta] = 0$$

From labor's FOC:

$$-\frac{\hat{C}^* v_L(1-N^*)}{1-\sigma} + v(1-N^*)F_N\left(\widehat{K}^*, N^*\right) = 0$$

$$\Rightarrow\ \frac{F_n\left(\widehat{K}^*, N^*\right)}{\hat{C}^*} - \frac{v_L(1-N^*)}{v(1-N^*)(1-\sigma)} = 0$$

From resource constraint:

$$\hat{C}^* = F\left(\widehat{K}^*, N^*\right) + (1 - \delta - \gamma)\widehat{K}^*$$

The steady state is characterized by these three equations.

Detrending the Growth Model (cont'd)

❖ Problem of the agent

$$\max_{\{C_t,K_{t+1},N_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left\{\frac{C_t^{1-\sigma}}{1-\sigma} v(1-N_t)\right\}, \qquad s.t. \begin{cases} C_t + K_{t+1} = F(K_t, \gamma^t N_t) + (1-\delta)K_t \\ K_0 > 0 \text{ given} \end{cases}$$

Let

$$\hat{C}_t = \frac{C_t}{\gamma^t}, \qquad \widehat{K}_t = \frac{K_t}{\gamma^t}$$

Note (due to constant returns to scale):

$$\frac{F(K_t, \gamma^t N_t)}{\gamma^t} = F(\widehat{K}_t, N_t)$$

The detrended version of the model becomes

$$\max_{\{\hat{C}_t,\widehat{K}_{t+1},N_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \underbrace{\tilde{\beta}^t}_{=[\beta\gamma^{1-\sigma}]^t} \left\{\frac{\hat{C}_t^{1-\sigma}}{1-\sigma} v(1-N_t)\right\}, \qquad s.t. \begin{cases} \hat{C}_t + \widehat{K}_{t+1} = F(\widehat{K}_t, N_t) + (1-\delta)\widehat{K}_t \\ K_0 > 0 \text{ given} \end{cases}$$

Steady state: $\{\hat{C}^*, \widehat{K}^*, N^*\}$ such that

$$(1) \quad -1 + \frac{\beta}{\gamma^\sigma}\left[F_k(\widehat{K}^*, N^*) + 1 - \delta\right] = 0$$

$$(2) \quad \frac{F_n(\widehat{K}^*, N^*)}{\hat{C}^*} - \frac{v_\ell(1-N^*)}{(1-\sigma)v(1-N^*)} = 0$$

$$(3) \quad \hat{C}^* = F(\widehat{K}^*, N^*) + (1-\delta-\gamma)\widehat{K}^*$$

Let $C_t^* = \hat{C}^*\gamma^t$, $K_t^* = \widehat{K}^*\gamma^t$, $Y^* = \hat{Y}^*\gamma^t = F(\widehat{K}^*, N^*)\gamma^t$

$$\Rightarrow \frac{Y_{t+1}^*}{Y_t^*} = \frac{C_{t+1}^*}{C_t^*} = \frac{K_{t+1}^*}{K_t^*} = \gamma$$

Note: $N_t = N^*$.

$$\frac{K_t^*}{Y_t^*} = \frac{\widehat{K}^*\gamma}{F(\widehat{K}^*, N^*)\gamma} = constant.$$

The ***balanced growth path*** is achieved when

(1) Output per hours worked grows at a constant rate
(2) The capital-output ratio is constant

- The balanced growth path and the steady state are just two sides of the same coin.

❖ Calibration.

➢ Assume some functional forms:

$$F(K,N) = K^\theta N^{1-\theta}$$
$$v(1-N) = (1-N)^{\alpha(1-\sigma)}$$

So the period utility becomes

$$\frac{[C_t(1-N_t)^{\alpha}]^{1-\sigma}}{1-\sigma}$$

The steady state conditions can be rewritten as

$$(1)\quad -1+\frac{\beta}{\gamma^{\sigma}}\left[\theta\left(\frac{\widehat{K}^*}{N^*}\right)^{\theta-1}+1-\delta\right]=0$$

$$(2)\quad (1-\theta)\frac{\left(\widehat{K}^*/N^*\right)^{\theta}}{\hat{C}^*}-\frac{\alpha(1-\sigma)(1-N^*)^{\alpha(1-\sigma)-1}}{(1-\sigma)(1-N^*)^{\alpha(1-\sigma)}}=0$$

$$(3)\quad \hat{C}^*=\widehat{K}^{*\theta}N^{*1-\theta}+(1-\delta-\gamma)\widehat{K}^*$$

- Decide on period length (what is a period: e.g. month, quarter, year, etc.)
- Pick a sample period (what period we take the data from: e.g. Q1:1964 – Q3:2010).
- Idea: pick parameter values so that the model's implications are consistent with the data over the selected sample period.
  - Unknowns: $\alpha,\beta,\delta,\gamma,\theta,\sigma$
  - If we use an alternative utility function:
    $$\ln C_t+\alpha\ln(1-N_t)$$
    Then we don't have to calibrate σ (or simply pick some σ, e.g. $\sigma=2$, for the original specification for utility).
  - $\boldsymbol{\gamma}$: Given Y^*_{t+1}/Y^*_t, pick γ to match the growth rate of GDP per capita
  - $\boldsymbol{\theta}$: given the production function $Y=F(K,N)=K^{\theta}(1-N)^{1-\theta}$ has the property
    $$\frac{F_kK}{Y}=\theta,\qquad \frac{F_nN}{Y}=1-\theta$$
    Assume factor of production are paid their marginal product. Then,
    $$\frac{F_kK}{Y}=\frac{income\ derived\ from\ capital}{total\ income}=\quad\theta$$
    $$\frac{F_nN}{Y}=\frac{income\ drived\ from\ labor}{total\ income}\quad=1-\theta$$
    - From the data, get the sample average for
      $$\frac{K_t}{Y_t},\qquad\frac{C_t}{Y_t},\qquad N_t$$
  - $\boldsymbol{\delta}$: given the two ratios, equation (3) implies the value of δ (just divide both sides of (3) by Y^*)
  - $\boldsymbol{\beta}$: then (1) is going to imply β (use the property of Cobb-Douglas function to get capital-output ratio)
  - $\boldsymbol{\alpha}$: using N^* and (2) we can calculate α.

- To study business cycles
  - Use this calibration
  - Assume that $Y_t=z_tF(K_t,\gamma^tN_t)$, and specify a stochastic process for $\{z_t\}$

Competitive Equilibrium (Decentralized Solution)

- Firms (owns only technology)
  - Maximize profits
  - Produce output using capital and labor as inputs
  - Do not own the factors of production
    - Firms owning only capital simplifies the problem, so that they don't face intertemporal trade-offs.
  - Production technology: $y_t = z_t F(k_t, n_t)$
    - Note that lower-case letters denote individual choices and capital letters denote aggregate values.
    - $\{z_t\}_{t=0}^{\infty}$ follows a stationary Markov process: $z'|z \sim \underbrace{Q(z,z')}_{\text{transition function}}$
    - z_t is common to all firms $\rightarrow$ there is no idiosyncratic risk, only aggregate risk.
    - F has the usual assumptions (cf supplementary notes)
  - Goods and factor markets are competitive. Normalize the price of output to 1.

- Problem of the Firm

$$\max_{k_t, n_t} z_t F(k_t, n_t) - \underbrace{r_t}_{\substack{\text{interest}\\\text{rate}}} k_t - \underbrace{w_t}_{\substack{\text{wage}\\\text{rate}}} n_t$$

  - r_t and w_t are going to the prices that clear the factor markets.
  - Aggregate state variables: K (aggregate capital stock), z
  - Factor prices: $r(K,z)$, $w(K,z)$

Rewrite the problem of the firm:

$$\max_{k,n} zF(k,n) - r(K,z)k - w(K,z)n$$

First-order conditions:

$$k: \quad zF_K(k,n) - r(K,z) = 0$$
$$n: \quad zF_N(k,n) - w(K,z) = 0$$

Since F is homogeneous of degree 1, F_K and F_N are homogeneous of degree 0. Then, the first-order conditions can be rewritten as

$$\left.\begin{aligned} zF_K\left(\frac{k}{n},1\right) &= r(K,z) \\ zF_N\left(\frac{k}{n},1\right) &= w(K,z) \end{aligned}\right\} \Rightarrow \frac{y}{n} \text{ is the same for all firms}$$

This implies that

- All firms choose the some k/n ratio.
- Therefore, the "size" (i.e. output) of a firm is indeterminate, because we only know the ratio of the two inputs, but not their levels.
- Number of firms is indeterminate.

By Euler's theorem for homogeneous functions:

$$F(k,n) = F_K(k,n)k + F_N(k,n)n \;\Rightarrow\; zF(k,n) = zF_K(k,n)k + zF_N(k,n)n$$

Use FOC's:

$$zF(k,n) = r(K,z)k + w(K,z)n \;\Rightarrow\; \text{firms make zero profits}$$

Thus, we focus on a single, representative (price-taking) firm.

$$\max_{K^f,N^f} zF(K^f,N^f) - r(K,z)K^f - w(K,z)N^f$$

where K^f is the demand for capital by the representative firm, and N^f is the demand for labor by the representative firm. FOC's:

$$K^f: \quad zF_K(K^f,N^f) - r(K,z) = 0$$
$$N^f: \quad zF_N(K^f,N^f) - w(K,z) = 0$$

In equilibrium, $K^f = K$ (market clearing condition for capital market).

- K is the aggregate capital stock, which is also the aggregate supply of capital by households.
- Note that capital is supplied inelastically by households

Also, in equilibrium, $N^f = N(K,z)$ (so that labor market clears)

- $N(K,z)$ is the aggregate labor supply by households given K, z

Thus, in equilibrium:

$$\begin{cases} zF_K\big(K, N(K,z)\big) = r(K,z) \\ zF_N\big(K, N(K,z)\big) = w(K,z) \end{cases}$$

- Households.
  - There is a continuum of (mass 1) ex-ante identical (same starting conditions and same preferences), infinitely-lived, households.
  - May own capital and are endowed with one unit of time per period.

  - Aggregate state variables: K, z
  - Individual state variable: k

- Problem of the household

$$V(k,K,z) = \max_{c,k',n}\{u(c,1-n) + \beta E[V(k',K',z')|z]\}$$

subject to

$$c + k' = r(K,z)k + w(K,z)n + (1-\delta)k$$
$$c, k' \geq 0$$
$$n \in [0,1]$$
$$K' = H(K,z), \qquad \text{[law of motion for aggregate capital]}$$
$$z'|z \sim Q(z,z')$$

  - Note that capital is supplied inelastically when interest rate is positive
  - Assume that depreciation is paid by the household

- A ***Recursive Competitive Equilibrium*** (***RCE***) is a list:

  - A value function (for households): $V(k,K,z)$
  - Individual decision rules: $c(k,K,z),\ h(k,K,z),\ n(k,K,z)$
  - Aggregate (per capita) policies: $C(K,z),\ H(K,z),\ N(K,z)$
  - Factor prices: $r(K,z),\ w(K,z)$

such that $\forall K, z$:

(1) Households maximizes utility, i.e.

$$\{c(k,K,z), h(k,K,z), n(k,K,z)\} = \arg\max_{c,k',n}\{u(c, 1-n) + \beta E[V(k', H(K,z), z')|z]\}$$

subject to

$$c + k' = r(K,z)k + w(K,z)n + (1-\delta)k$$
$$c, k' \geq 0$$
$$z'|z \sim Q(z,z')$$

and

$$V(k,K,z) = u\big(c(k,K,z), 1 - n(k,K,z)\big) + \beta E[V(h(k,K,z), H(K,z), z')|z], \qquad \forall k$$

(2) Firms maximize profits, i.e.

$$r(K,z) = zF_K\big(K, N(K,z)\big)$$
$$w(K,z) = zF_N\big(K, N(K,z)\big)$$

- Note that the market clearing conditions in the factor markets is (implicitly) embedded in these two equations, which imposes, for example, $K^f(K,z) = K$ and $N^f(K,z) = N(K,z)$. This way, we don't have to worry of clearing these two markets later on.

(3) Consistency (sum of individual choices equal the aggregate):

$$c(K,K,z) = C(K,z)$$
$$h(K,K,z) = H(K,z)$$
$$n(K,K,z) = N(K,z)$$

(4) Market clearing (in the goods market):

$$C(K,z) + H(K,z) = zF\big(K, N(K,z)\big) + (1-\delta)K$$

RBC Model

- In applications, we assume:
  - Technology:

$$Y_t = z_t K_t^{\theta} (\gamma^t N_t)^{1-\theta}, \qquad \gamma \geq 1,\ \theta \in [0,1]$$
$$\ln z_t = \rho \ln z_{t-1} + \epsilon_t, \qquad \epsilon \sim (0, \sigma_\epsilon^2)$$

    - Note that the measure of z_t is model-dependent. Change a production function, the values of z_t will change.
  - Preferences:

$$u(c_t, \ell_t) = \ln(c_t) + \underbrace{\frac{\alpha(1-n_t)^{1-\chi}}{1-\chi}}_{v(\ell)}, \qquad \alpha > 0,\ \chi \geq 0$$

    - χ governs the volatility of hours
  - Period length: 1 quarter.
  - Sample period: post-war developed economies

- Benchmark case: only shocks to z_t
  (1) Can explain about $2/3$ of the volatility in GDP per capita.
  (2) Explains (more or less) well consumption and investment behaviors; however, consumption is too smooth relative to the data.
    - One way to make consumption more volatile is to introduce frictions in the model
  (3) Hours worked are smoother (or less volatile) than in the data
    - Can use χ to mitigate this problem, or incorporate labor market frictions
  (4) Average labor productivity (Y_t/N_t) and total hours (N_t) are highly and positively correlated in the model; whereas in the data, this correlation is either zero or slightly negative.
    - To fix this problem, again, need to introduce friction or heterogeneity, etc.

- Extensions / Variations
  (1) Impulses, e.g.
    - Shocks to taxes or government expenditure (reduces the role of z_t)
    - Energy price fluctuations
    - Monetary policy
  (2) Propagation mechanisms, e.g.
    - "time-to-build" $\rightarrow$ investment takes some time to mature
    - Adjustment costs in investment
    - Indivisible labor
    - Monopolistic competition
    - Variable capital utilization
    - Habit persistence in consumption
    - Asymmetric information
    - Labor market frictions, e.g. search costs
    - News (information) shocks

Extension and Re-interpretation

- Indivisible labor. Suppose

$$n_t = \begin{cases} \bar{n} & \text{work} \\ 0 & \text{not work} \end{cases}$$

  - Labor pays a wage w per hour worked.
  - Let λ be the probability of working
  - Consider an insurance market for employment status.
    - A contract offers 1 unit of consumption if agent does not work and 0 otherwise.
    - Let q be the amount of insurance purchased, and p be the price of a contract.
    - Insurance market is perfectly competitive $\Rightarrow$ insurance firms make zero expected profits on each contract

  - Agent's problem.

$$\max_q \lambda[u(c_W) + v(1-\bar{n})] + (1-\lambda)[u(c_N) + v(1)]$$

    where

$$c_W = w\bar{n} - qp$$
$$c_N = q(1-p)$$

    First-order condition w.r.t. q:

$$-\lambda u_c(c_W)p + (1-\lambda)u_c(c_N)(1-p) = 0$$

    Since insurance firms make zero expected profit,

$$\lambda p + (1-\lambda)(p-1) = 0 \;\Rightarrow\; p = 1-\lambda$$

    Plug $p = 1-\lambda$ into FOC:

$$-\lambda(1-\lambda)u_c(c_W) + (1-\lambda)\lambda u_c(c_N) = 0$$
$$\Rightarrow\; u_c(c_W) = u_c(c_N) \;\Rightarrow\; c_W = c_N = c$$
$$\Rightarrow\; w\bar{n} - qp = q(1-p)$$
$$\Rightarrow\; w\bar{n} - q(1-\lambda) = q\lambda$$
$$\Rightarrow\; \boxed{q = w\bar{n}}$$
$$\Rightarrow\; \boxed{c = w\bar{n}\lambda}$$

    - Note that the separability of consumption and leisure preferences is important in deriving this result.

  - Consider an agent choosing λ (probability of finding a job)

$$\max_\lambda u(w\bar{n}\lambda) + \lambda v(1-\bar{n}) + (1-\lambda)v(1)$$

    FOC:

$$u_c(w\bar{n}\lambda)w\bar{n} + \underbrace{v(1-\bar{n}) - v(1)}_{\equiv A<0} = 0$$
$$u_c(w\bar{n}\lambda)w\bar{n} + A = 0$$

  - Consider a planner that solves the following problem:

$$\max_{c_W, c_N, \lambda} \lambda[u(c_W) + v(1-\bar{n})] + (1-\lambda)[u(c_N) + v(1)]$$

    subject to

$$\underbrace{\lambda c_W}_{\substack{\text{fraction}\\ \text{of workers}}} + \underbrace{(1-\lambda)c_N}_{\substack{\text{fraction of}\\ \text{non-worker}}} = \underbrace{\lambda w\bar{n}}_{\substack{\text{income generated}\\ \text{by workers}}}$$

FOC's (μ is the Lagrange multiplier):

$$\left.\begin{array}{rr} c_W: & \lambda u_c(c_W) - \mu\lambda = 0 \\ c_N: & (1-\lambda)u_c(c_N) - \mu(1-\lambda) = 0 \end{array}\right\} \Rightarrow \; u_c(c_W) = u_c(c_N)$$

$$\Rightarrow \; c_W = c_N = c$$

$$\Rightarrow \; c = \lambda w\bar{n}, \qquad \text{[substitute previous line to constraint]}$$

Use this result to simplify the problem of the planner:

$$\max_{\lambda} u(\lambda w\bar{n}) + \lambda v(1-\bar{n}) + (1-\lambda)v(1)$$

which is the same as the agent's problem $\Rightarrow$ same solution as the agent.

- This shows that the agent's solution is the first best.

- Consider the problem of an agent with (infinitely) divisible labor choice, with linear disutility of working:

$$\max_{h} u(c) - \alpha h, \qquad s.t.\; c = (w\bar{n})h$$

$$\Rightarrow \; \max_{h} u(w\bar{n}h) - \alpha h$$

FOC:

$$u_c(w\bar{n}h)w\bar{n} - \alpha = 0$$

- This is the same equation that governs the agent's labor decision in most models (with or without capital).
- Let $\alpha = A$. Then,

$$u_c(w\bar{n}h)w\bar{n} + A = 0 \;\; \Rightarrow \;\; h = \lambda$$

- This shows that, with perfect insurance market, the solution of indivisible labor problem is equivalent to the divisible labor problem when labor preference is linear.

Home (or Household) Production

- Observations (from surveys):
  - People spend about 25% of discretionary time in household activities (e.g. cleaning, cooking, etc.)
  - People spend about 33% of discretionary time is spent working for paid compensation, i.e. market activities.
  - The rest would be considered leisure time.
  - Investment in household capital (e.g. consumer durables and residential) is greater than investment in market capital (e.g. machines, non-residential estates)
  - Home output is about 25 – 50% of GNP (depending on various ways of measuring)

- Model: Problem of the household

$$\max_{\{c_{Mt},c_{Ht},k_{M(t+1)},k_{H(t+1)},n_{Mt},n_{Ht}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_{Mt}, c_{Ht} n_{Mt}, n_{Ht})$$

subject to

$$c_{Mt} + k_{M(t+1)} + k_{H(t+1)} = r_t k_{Mt} + w_t n_{Mt} + (1 - \delta_M) k_{Mt} + (1 + \delta_H) k_{Ht}$$
$$c_{Ht} = g(k_{Ht}, n_{Ht})$$
k_{M0}, k_{H0} given
$$\ell_t = 1 - n_{Mt} - n_{Ht}$$
non-negativity constraints

  - Note that while home production is not marketed, capital for home production is.
  - Assumptions on g

$$g_k > 0, \qquad g_n > 0, \qquad g_{kk} < 0, \qquad g_{nn} < 0$$

Define another function

$$v(c_M, n_M, k_H) \equiv \max_{n_H} u\left(c_M, \underbrace{g(k_H, n_H)}_{c_H}, n_M, n_H\right)$$

Hence, the problem can be rewritten as

$$\max_{\{c_{Mt},n_{Mt},k_{M(t+1)},k_{H(t+1)}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t v(c_{Mt}, n_{Mt}, k_{Ht})$$

subject to

$$c_{Mt} + k_{M(t+1)} + k_{H(t+1)} = r_t k_{Mt} + w_t n_{Mt} + (1 - \delta_M) k_{Mt} + (1 - \delta_H) k_{Ht}$$
k_{M0}, k_{H0} given
non-negativity constraints

  - Hence, the problem becomes one where one can interpret it as households having preference for an extra consumption good call “home capital”.

Optimal Policy

- Environment:
  - Continuum of agents (mass 1) [can also start with representative agent, same result]
  - Technology: $y_t = A_t n_t$
    - Note that we abstract from capital, and the technology is constant returns to scale
    - $\{A_t\}_{t=0}^{\infty}$ given, (i.e. known in $t = 0$).
  - Preferences: $u(c, 1-n)$, with standard assumptions and $u_{c\ell} = 0$ (consumption and leisure are separable in u)

  - Government: needs to finance a given sequence of expenditure $\{G_t\}_{t=0}^{\infty}$ (known at $t = 0$)
    - Use taxes on income τ_t (in this model, income and production are the same thing)
    - Can issue one period bonds
    - Notations:
      - Let B_t be the stock of bonds at the beginning of t (which mature in t); B_{t+1} is the stock of bonds issued in t that mature in $t+1$.
      - Let q_t be the price of a bond that pays 1 unit of consumption in $t+1$ (discount price). So q_t measures how much one is willing to pay today for 1 unit of consumption tomorrow.
      - Initial stock of bonds, B_0.
    - Government budget constraint (per period): at each t

$$\underbrace{G_t + B_t}_{\substack{\text{total expenditure} \\ \text{in } t}} = \underbrace{\tau_t Y_t + q_t B_{t+1}}_{\substack{\text{total revenue} \\ \text{in } t}}$$

    where $Y_t = A_t N_t$, N_t being the aggregate labor

  - Problem of the agent. Given $\{\tau_t, q_t\}_{t=0}^{\infty}$, the agent solves the problem:

$$\max_{\{c_t, n_t, b_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t)$$

    subject to

$$c_t + q_t b_{t+1} = (1-\tau) A_t n_t + b_t, \qquad \forall t = 0,1,\ldots$$
$$b_0 \text{ given}$$

    - Even though there is no capital, there is still dynamics in the economy because there are bonds. This shows two channels via which we can introduce dynamics into our models.
    - Note that without capital, having a firm which rents labor from the agents or letting agents own firms and producing themselves are completely equivalent. This is not the case if there is capital (since constant returns to scale happens to both k and n at the same time).

    First-order conditions (use $c_t = (1-\tau_t)A_t n_t + b_t - q_t b_{t+1}$ for simpler notations):

$$\left.\begin{aligned} n_t: &\quad \beta^t u_c(c_t)(1-\tau_t)A_t - \beta^t u_\ell(1-n_t) = 0 \\ b_{t+1}: &\quad -\beta^t u_c(c_t) q_t + \beta^{t+1} u_c(c_{t+1}) = 0 \\ &\quad c_t = (1-\tau_t)A_t n_t + b_t - q_t b_{t+1} \end{aligned}\right\}, \qquad \forall t = 0,1,\ldots$$

Transverality condition:

$$\lim_{t\to\infty} \beta^t u_c(c_t) q_t b_{t+1} = 0$$

and b_0 given.

- In equilibrium, given $b_0 = B_0$:

$$\begin{matrix} c_t = C_t \\ n_t = N_t \end{matrix} \quad \text{and} \quad b_{t+1} = B_{t+1}$$

We now have

$$\begin{aligned} &(1) \quad u_c(C_t)(1-\tau_t)A_t - u_\ell(1-N_t) = 0 \\ &(2) \quad -u_c(C_t)q_t + \beta u_c(C_{t+1}) = 0 \\ &(3) \quad C_t = (1-\tau_t)A_t N_t + B_t - q_t B_{t+1} \\ &(4) \quad G_t + B_t = \tau_t A_t N_t + q_t B_{t+1} \end{aligned}$$

Rewrite (4) as

$$B_t - q_t B_{t+1} = \tau_t A_t N_t - G_t$$

Substitute this into (3):

$$C_t = (1-\tau_t)A_t N_t + \tau_t A_t N_t - G_t$$
$$\Rightarrow \boxed{C_t + G_t = A_t N_t}$$

This is the resource constraint.

- Note that this differs from the standard model in that all goods produced (RHS) is completely spent on either consumption or financing government debt; whereas in the standard model, where there is capital, we have capital accumulation (i.e. K_{t+1})
- The instruments, τ_t and B_t, will decide how much weight is put on C_t and G_t.

From (2), we have the price of bonds at t:

$$q_t = \beta \frac{u_c(C_{t+1})}{u_c(C_t)}$$

- Bench mark cases.
  - Case I (no supply of bonds): $B_t = 0$ for all $t = 0,1,\dots$

    $\{C_t, N_t, \tau_t\}_{t=0}^{\infty}$ satisfy

$$\begin{aligned} &(5) \quad u_c(C_t)(1-\tau_t)A_t - u_\ell(1-N_t) = 0 \\ &(6) \quad G_t = \tau_t A_t N_t \\ &(7) \quad C_t + G_t = A_t N_t \end{aligned}$$

for all $t = 0,1,\dots.$

From (6), we have

$$\tau_t = \frac{G_t}{A_t N_t}$$

Plug it into (5), we have

$$u_c(C_t)\left[A_t - \frac{G_t}{A_t N_t}A_t\right] - u_\ell(1-N_t) = 0$$
$$\Rightarrow \; u_c(C_t)\frac{[A_t N_t - G_t]}{N_t} - u_\ell(1-N_t) = 0$$

Use (7),

$$u_c(C_t)C_t - u_\ell(1-N_t)N_t = 0$$

Given $\{G_t, A_t\}_{t=0}^{\infty}$, $\{C_t, N_t\}_{t=0}^{\infty}$ solves

$$u_c(C_t)C_t - u_\ell(1-N_t)N_t = 0$$
$$C_t + G_t = A_t N_t$$

for all $t = 0,1, \ldots.$

- Example (assume a functional form for u):

  $$u(C, 1-N) = \ln C + \alpha(1-N), \qquad \alpha > 1$$

  Then,

  $$(8) \quad \frac{1}{C_t}C_t - \alpha N_t = 0 \;\Rightarrow\; N_t = \frac{1}{\alpha}$$

  This implies for (7) that

  $$C_t + G_t = \frac{A_t}{\alpha} \;\Rightarrow\; C_t = \frac{A_t}{\alpha} - G_t$$

  And for (6),

  $$\tau_t = \frac{\alpha G_t}{A_t}$$

  - Comparative statics:

    $\uparrow G_t \;\Rightarrow\; \downarrow C_t$ (one-for-one)
    $\Rightarrow\; \uparrow \tau_t$
    $\Rightarrow$ no change in N_t

    $\uparrow A_t \;\Rightarrow\; \uparrow C_t$
    $\Rightarrow\; \downarrow \tau_t$
    $\Rightarrow$ no change in N_t

    All effects are felt in t because there is intertemporal trade-offs (note that no supply of bonds basically rids the model of dynamics).

Optimal Policy (cont'd)

- Problem of the agent (cont'd).
  Given $\{\tau_t, q_t\}_{t=0}^{\infty}$, the agent solves

$$\max_{\{c_t, b_{t+1}, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t)$$

  subject to

$$c_t + q_t b_{t+1} = (1-\tau_t)A_t n_t + b_t, \qquad \forall t = 0,1,\ldots$$

  with b_0 and $\{A_t\}_{t=0}^{\infty}$ given.

- Competitive Equilibrium.
  - Given B_0, $\{A_t, G_t, B_{t+1}\}_{t=0}^{\infty}$, a competitive equilibrium is a sequence $\{C_t, N_t, q_t, \tau_t\}_{t=0}^{\infty}$ such that

$$\left.\begin{aligned} &(1) \quad u_c(C_t)(1-\tau_t)A_t - u_\ell(1-N_t) = 0 \\ &(2) \quad C_t + G_t = A_t N_t \\ &(3) \quad q_t = \frac{\beta u_c(C_{t+1})}{u_c(C_t)} \\ &(4) \quad G_t + B_t = \tau_t A_t N_t + q_t B_{t+1} \end{aligned}\right\}, \qquad \forall t = 0,1,\ldots$$

    and the TVC

$$(5) \quad \lim_{t\to\infty} \beta^t u_c(C_t) q_t B_{t+1} = 0.$$

    - Note that we could alternatively take $\{\tau_t\}_{t=0}^{\infty}$ as given, and use $\{B_t\}_{t=0}^{\infty}$ in the definition of the CE.

  - Take (4): $G_t + B_t = \tau_t A_t N_t + q_t B_{t+1}$, $\forall t = 0,1,\ldots$. Sum over all periods:

$$\sum_{t=0}^{\infty}[G_t + B_T] = \sum_{t=0}^{\infty}[\tau_t A_t N_t + q_t B_{t+1}]$$

$$\sum_{t=0}^{\infty}\beta^t u_c(C_t)[G_t + B_T] = \sum_{t=0}^{\infty}\beta^t u_c(C_t)[\tau_t A_t N_t + q_t B_{t+1}]$$

$$\Rightarrow \quad (***) \quad 0 = \sum_{t=0}^{\infty}\beta^t u_c(C_t)[\tau_t A_t N_t - G_t] + \underbrace{\sum_{t=0}^{\infty}\beta^t u_c(C_t) q_t B_{t+1} - \sum_{t=0}^{\infty}\beta^t u_c(C_t) B_t}_{(*)}$$

    Use (3) on (*):

$$(*) = \sum_{t=0}^{\infty}\beta^t u_c(C_t)\cdot\frac{\beta u_c(C_{t+1})}{u_c(C_t)}\cdot B_{t+1} - \sum_{t=1}^{\infty}\beta^t u_c(C_t)B_t - u_c(C_0)B_0$$

$$= \sum_{t=0}^{\infty}\beta^{t+1} u_c(C_{t+1})B_{t+1} - \sum_{t=1}^{\infty}\beta^t u_c(C_t)B_t - u_c(C_0)B_0$$

    Use (3) on (5):

$$(**) \quad \lim_{t\to\infty}\beta^{t+1}u_c(C_{t+1})B_{t+1} = 0$$

Therefore, we have (*) converging to

$$(*) = \underbrace{\sum_{t=0}^{\infty} \beta^{t+1} u_c(C_{t+1}) B_{t+1} - \sum_{t=1}^{\infty} \beta^t u_c(C_t) B_t}_{\to 0 \text{ due to } (**)} - u_c(C_0) B_0$$
$$= -u_c(C_0) B_0$$

Then, $(***)$ becomes

$$(***)' \quad 0 = \sum_{t=0}^{\infty} \beta^t u_c(C_t)[\tau_t A_t N_t - G_t] - u_c(C_0) B_0$$

Use (1) to get

$$u_c(C_t) A_t - u_c(C_t) \tau_t A_t - u_\ell(1 - N_t) = 0$$
$$\Rightarrow \quad (1)' \quad u_c(C_t) \tau_t A_t = u_c(C_t) A_t - u_\ell(1 - N_t)$$

Then $(***)'$ becomes

$$0 = \sum_{t=0}^{\infty} \beta^t \left[\underbrace{u_c(C_t) \tau_t A_t}_{\text{use } (1)'} N_t - u_c(C_t) G_t \right] - u_c(C_0) B_0$$
$$= \sum_{t=0}^{\infty} \beta^t [u_c(C_t) A_t N_t - u_\ell(1 - N_t) N_t - u_c(C_t) G_t] - u_c(C_0) B_0$$

From (2): $C_t = A_t N_t - G_t$

$$\boxed{0 = \sum_{t=0}^{\infty} \beta^t [u_c(C_t) C_t - u_\ell(1 - N_t) N_t] - u_c(C_0) B_0}$$

This is the ***Implementability Constraint***. (2) is also a constraint:

$$\boxed{C_t + G_t = A_t N_t}, \qquad \forall t = 0,1, \ldots$$

- Problem of a benevolent government:

$$\max_{\{C_t, N_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t, 1 - N_t)$$

subject to

$$C_t + G_t = A_t N_t, \qquad \forall t = 0,1, \ldots$$
$$\sum_{t=0}^{\infty} \beta^t [u_c(C_t) C_t - u_\ell(1 - N_t) N_t] - u_c(C_0) B_0 = 0$$
$$B_0 \text{ given}$$

Optimal Policy (with Debt) (cont'd)

❖ Recall the equations we derived last time:

$$(1)\quad u_c(C_t)(1-\tau_t)A_t - u_\ell(1-N_t)$$
$$(2)\quad C_t + G_t = N_t$$
$$(3)\quad q_t = \frac{\beta u_c(C_{t+1})}{u_c(C_t)}$$
$$(4)\quad B_t + G_t = \tau_t A_t N_t + q_t B_{t+1}$$
$$(5)\quad \lim_{t\to\infty} \beta^t u_c(C_t) q_t B_{t+1} = 0$$

- Agent's budget constraint:
$$c_t + q_t b_{t+1} = (1-\tau_t)A_t n_t + b_t$$
In equilibrium, we have the resource constraint
$$C_t + q_t B_{t+1} = (1-\tau_t)A_t N_t + B_t$$

❖ Problem of the (benevolent) government

$$\max_{\{B_{t+1},\tau_t,C_t,N_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t, 1-N_t)$$

subject to

$$B_t + G_t = \tau_t A_t N_t + \overbrace{\left(\frac{\beta u_c(C_{t+1})}{u_c(C_t)}\right)}^{q_t} B_{t+1}$$
$$C_t + G_t = A_t N_t$$
$$u_c(C_t)A_t(1-\tau_t) - u_\ell(1-N_t) = 0$$
$$B_0 \text{ given}$$
$$\lim_{t\to\infty} \beta^{t+1} u_c(C_{t+1}) B_{t+1} = 0$$

for all $t = 0,1,\ldots.$

- The government is maximizing social welfare subject to that agents behave competitively.

- Last time, using (1) – (5), we derived the ***implementability constraint***:
$$\sum_{t=0}^{\infty} \beta^t [u_c(C_t)C_t - u_\ell(1-N_t)N_t] - u_c(C_0)B_0 = 0$$
- This simplifies the problem of the government to the following:
$$\max_{\{C_t,N_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t, 1-N_t)$$
subject to
$$C_t + G_t = A_t N_t, \qquad \forall t \in \{0\} \cup \mathbb{N}$$
$$\sum_{t=0}^{\infty} \beta^t [u_c(C_t)C_t - u_\ell(1-N_t)N_t] - u_c(C_0)B_0 = 0$$
$$B_0 \text{ given}$$
    - Note that with the first constraint, we can further reduce the number of choice

variables.

- Let: $u(C_t, 1-N_t) = \ln C_t + \alpha(1-N_t)$.
  - The implementability constraint becomes
$$\sum_{t=0}^{\infty}\beta^t\left[\frac{1}{C_t}C_t - \alpha N_t\right] - \frac{B_0}{C_0} = 0$$
Use $C_t + G_t = A_t N_t$:
$$\sum_{t=0}^{\infty}\beta^t\left[1 - \frac{\alpha}{A_t}C_t - \frac{\alpha}{A_t}G_t\right] - \frac{B_0}{C_0} = 0$$
  - The problem of the government becomes
$$\max_{\{C_t\}_{t=0}^{\infty}}\sum_{t=0}^{\infty}\beta^t\left[\ln C_t + \alpha\left(1 - \frac{C_t + G_t}{A_t}\right)\right]$$
subject to
$$\sum_{t=0}^{\infty}\beta^t\left[1 - \frac{\alpha}{A_t}C_t - \frac{\alpha}{A_t}G_t\right] - \frac{B_0}{C_0} = 0$$
$$B_0 \text{ given}$$
Let $\alpha_t \equiv \alpha/A_t$. When taking first-order conditions, take note of the difference between the initial period and the other periods.
    - Here we're also assuming that government commits to its policy decision at $t = 0$.

Let λ be the (only) Lagrange multiplier. The FOC's are
$$C_0: \quad \underbrace{\frac{1}{C_0} - \alpha_0}_{u_c - u_\ell} + \lambda\left[-\alpha_0 + \frac{B_0}{C_0^2}\right] = 0$$
$$C_t: \quad \beta^t\left[\frac{1}{C_t} - \alpha_t\right] - \lambda\beta^t\alpha_t = 0, \qquad \forall t = 1,2,\dots$$
$$\Rightarrow \lambda = \frac{1}{\alpha_t C_t} - 1 \Leftrightarrow \underbrace{\alpha_t C_t}_{\alpha\frac{C_t}{A_t}} \text{ is constant } \forall t \geq 1$$

    - Note that
$$\frac{1}{C_0} - \alpha_0 = \left[\frac{1}{\alpha_t C_t} - 1\right]\left[\alpha_0 - \frac{B_0}{C_0^2}\right]$$
$$\Rightarrow (6) \quad \left(\frac{1}{\alpha_0 C_0} - 1\right) = \left(\frac{1}{\alpha_t C_t} - 1\right)\left(1 - \frac{B_0}{C_0}\cdot\frac{1}{\alpha_0 C_0}\right)$$
$$(7) \quad \sum_{t=0}^{\infty}\beta^t[1 - \alpha_t C_t - \alpha_t G_t] - \frac{B_0}{C_0} = 0$$
The latter two equations characterize the solution of the government's problem.

  - Assume $B_0 = 0$. Then,

$$\alpha_0 C_0 = \alpha_t C_t \;\Rightarrow\; \frac{\alpha}{A_0} C_0 = \frac{\alpha}{A_t} C_t \;\Rightarrow\; C_t = \frac{A_t}{A_0} C_0$$

(7) implies that

$$\frac{1-\alpha_t C_t}{1-\beta} - \sum_{t=0}^{\infty} \beta^t \alpha_t C_t = 0 \;\Rightarrow\; 1 - \alpha_t C_t = \underbrace{(1-\beta)\sum_{t=0}^{\infty} \beta^t \alpha_t C_t}_{g}$$

$$\Rightarrow\; C_t = \frac{1-g}{\alpha_t}$$

$$\Rightarrow\; \boxed{C_t = A_t \frac{1-g}{\alpha}}, \qquad \forall t \geq 0$$

The implied labor decision is

$$N_t = \frac{C_t + G_t}{A_t} \;\Rightarrow\; \boxed{N_t = \frac{1-g}{\alpha} + \frac{G_t}{A_t}}$$

- What about policy?
  Use (1):

$$\frac{1}{C_t} A_t (1-\tau_t) - \alpha = 0 \;\Rightarrow\; 1 - \tau_t = \frac{\alpha C_t}{A_t} (= \alpha_t C_t)$$

$$\Rightarrow\; 1 - \tau_t = 1 - g$$

$$\Rightarrow\; \boxed{\tau_t = g}, \qquad \forall t \geq 0$$

So, constant tax rate, i.e. tax-smoothing.

Use (3) to get the price of bonds:

$$q_t = \frac{\beta C_t}{C_{t+1}} \;\Rightarrow\; \boxed{q_t = \frac{\beta A_t}{A_{t+1}}}$$

Use (4) to get the sequence of debts

$$G_t + B_t = \tau_t A_t N_t + q_t B_{t+1} \;\Rightarrow\; G_t + B_t = g[C_t + G_t] + \frac{\beta A_t}{A_{t+1}} B_{t+1}$$

$$\Rightarrow\; (1-g) G_t - B_t = \frac{g(1-g)A_t}{\alpha} + \frac{\beta A_t}{A_{t+1}} B_{t+1}$$

$$\Rightarrow\; B_{t+1} = \frac{A_{t+1}}{\beta A_t}\left[(1-g)\left[G_t - \frac{g A_t}{\alpha}\right] + B_t\right]$$

  - Since tax rate is constant, debt has to absorb the temporary variation in the government expenditure. It is better to use debt than taxes because agents prefer consumption smoothing, and too much changes in tax rates will mess up the consumption smoothing. However, since debt is just a saving instrument, agents can use debt to smooth out variations in the economy.

Optimal Policy (cont'd)

- Planner's Problem

$$\max_{\{C_t, N_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t, 1 - N_t)$$

subject to feasibility/resource constraint:

$$C_t + G_t = A_t N_t, \qquad \forall t = 0,1,\ldots$$

The solution is characterized by $\{C_t, N_t\}_{t=0}^{\infty}$ such that

$$u_c(C_t)A_t = u_\ell(1 - N_t)$$
$$C_t + G_t = A_t N_t$$

for all $t = 0,1,\ldots.$

- If $u(C, 1 - N) = \ln C + \alpha(1 - N)$, then

$$\frac{A_t}{C_t} = \alpha \;\Rightarrow\; C_t = \frac{A_t}{\alpha}$$
$$\Rightarrow\; N_t = \frac{1}{\alpha} - \frac{G_t}{A_t}$$

So when utility is linear in leisure, labor is going to absorb all the variations in G_t, while productivity shocks affects only consumption.

- Competitive Equilibrium with no debt is characterized by $\{C_t, N_t, \tau_t\}_{t=0}^{\infty}$ such that

$$u_c(C_t)A_t(1 - \tau_t) = u_\ell(1 - N_t)$$
$$C_t + G_t = A_t N_t$$
$$G_t = \tau_t A_t N_t$$

for all $t = 0,1,\ldots.$

From the last equation, derive tax as

$$\tau_t = \frac{G_t}{A_t N_t}$$

and sub into the first equation

$$u_c(C_t)A_t\left[1 - \frac{G_t}{A_t N_t}\right] = u_\ell(1 - N_t)$$
$$\Rightarrow \begin{cases} u_c(C_t)C_t = u_\ell(1 - N_t)N_t \\ C_t + G_t = A_t N_t \end{cases}$$

- Assuming the same utility form as in the planner's problem, we have

$$\frac{1}{C_t} C_t = \alpha N_t \;\Rightarrow\; N_t = \frac{1}{\alpha}$$
$$C_t = \frac{A_t}{\alpha} - G_t$$

So as opposed to the first best case, the competitive equilibrium (in which taxes are distortionary), consumption absorbs all the variations in *both* A_t and G_t.

- Optimal policy with taxes and debt

- Recall the CE (with debt) is

$$u_c(C_t)(1-G_t)A_t = u_\ell(1-N_t)$$
$$C_t + G_t = A_t N_t$$
$$q_t = \frac{\beta u_c(C_{t+1})}{u_c(C_t)}$$
$$B_t + G_t = \tau_t A_t N_t + q_t B_{t+1}$$
$$\lim_{t\to\infty} \beta^t u_c(C_t) q_t B_{t+1} = 0$$

- Implementability constraint is

$$\sum_{t=0}^{\infty} \beta^t [u_c(C_t)C_t - u_\ell(1-N_t)N_t] - u_c(C_0)B_0 = 0$$

  - Additional constraint
  $$C_t + G_t = A_t N_t, \qquad \forall t = 0,1,\ldots$$
  - Compare with the no debt case, instead of satisfying the constraint
  $$u_c(C_t)C_t - u_\ell(1-N_t)N_t$$
  every period, debt allows the benevolent government to just satisfy this constraint "on average" (i.e. a weighted average by β). So debt can be used to absorb some of the variations.

❖ Let $\{C_t^*, N_t^*\}_{t=0}^{\infty}$ be the first-best (FB) allocation. In particular, the sequence satisfy

$$u_c(C_t^*)A_t = u_\ell(1-N_t^*)$$
$$C_t^* + G_t = A_t N_t^*$$

Can the FB be implemented in a CE with debt and distortionary taxes?

- At the FB,

$$\sum_{t=0}^{\infty} \beta^t \left[u_c(C_t^*)C_t^* - u_c(C_t^*) \underbrace{A_t N_t^*}_{C_t^*+G_t} \right] - u_c(C_0^*)B_0 = 0$$

$$-\sum_{t=0}^{\infty} \beta^t u_c(C_t^*)G_t - u_c(C_0^*)B_0 = 0$$

$$B_0 = \frac{-1}{u_c(C_0^*)} \sum_{t=0}^{\infty} \beta^t u_c(C_t^*)G_t < 0$$

  - If the government wants to implement the FB, it has to start with a large enough amount of savings (negative debt), so that it does not need to use distortionary taxes to finance G_t. So what the government's tax smoothing is in fact "distortion smoothing"
  - This is called the "war chest".

Optimal Capital and Labor Taxes

- Environment—a variant of the standard neoclassical growth model
  - Preferences: $u(c, \ell)$ with standard assumption, and additively separable in c and ℓ
  - Technology: $y_t = z_t F(k_t, n_t)$, where F satisfies the usual assumptions, and $\{z_t\}_{t=0}^{\infty}$ is known at $t = 0$.
  - Government
    - Benevolent, and can commit to future policy choices
    - $\{G_t\}_{t=0}^{\infty}$ is given at $t = 0$, with $G_t \in [\underline{G}, \overline{G}]$, $0 \leq \underline{G} < \overline{G} < \infty$.
    - Instruments to finance G_t
      - Capital income taxes $\rightarrow$ no capital depreciation tax allowance (for simplicity)
      - Labor income taxes
        - It's important to have a *complete tax system*, i.e. a tax system that targets all the trade-off margins that a planner faces. In this setting, two tax instruments would suffice, namely, one for the consumption-leisure trade-off (labor income tax) and one for intertemporal consumption trade-off (capital income tax).
      - One-period bonds
    - Government Budget Constraint (GBC)
$$B_t + G_t = \tau_t^K r_t K_t + \tau_t^N w_t N_t + q_t B_{t+1}, \qquad \forall t = 0,1,\ldots$$
      - Suppose tax allowance is made for capital depreciation, the government simply collects capital taxes on $(r_t - \delta)K_t$ instead of $r_t K_t$.

  - Problem of the agent. Given $\{r_t, w_t, q_t, \tau_t^K, \tau_t^N\}_{t=0}^{\infty}$, the agent solves
$$\max_{\{c_t, k_{t+1}, b_{t+1}, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t)$$
subject to
$$\begin{aligned} c_t + k_{t+1} + q_t b_{t+1} &= (1 - \tau_t^K) r_t k_t + (1 - \tau_t^N) n_t + (1 - \delta) k_t + b_t \\ c_t, k_{t+1} &\geq 0 \\ n_t &\in [0,1] \\ k_0, b_0 &\text{ given} \end{aligned}$$
FOC's:
$$k_{t+1}: \quad -u_c(c_t) + \beta u_c(c_{t+1}) \left[\overbrace{(1 - \tau_t^K) r_{t+1}}^{\tilde{r}_{t+1}} + 1 - \delta \right] = 0$$
$$n_t: \quad u_c(c_t) \underbrace{(1 - \tau_t^N) w_t}_{\tilde{w}_t} - u_\ell(1 - n_t) = 0$$
$$b_{t+1}: \quad -q_t u_c(c_t) + \beta u_c(c_{t+1}) = 0$$
$$c_t = (1 - \tau_t^K) r_t k_t + (1 - \tau_t^N) w_t n_t + (1 - \delta) k_t + b_t - k_{t+1} - q_t b_{t+1}$$
$$TVC_k \quad \lim_{t \to \infty} \beta^t u_c(c_t) k_{t+1} = 0$$
$$TVC_b \quad \lim_{t \to \infty} \beta^t u_c(c_t) q_t b_{t+1} = 0$$
for all $t = 0,1, \ldots.$

- Optimal solutions to agent's problem (using aggregate variables).
  Given (K_0, B_0), $\{z_t, G_t\}_{t=0}^{\infty}$, and $\{\tau_t^K, \tau_t^N, B_{t+1}\}_{t=0}^{\infty}$, the optimal solutions is a sequence $\{C_t, K_{t+1}, N_t, q_t\}_{t=0}^{\infty}$ that solves
  (1) $u_c(C_t) + \beta u_c(C_{t+1})[(1-\tau_t^K)z_t F_K(K_{t+1}, N_{t+1}) + 1 - \delta] = 0$
  (2) $u_c(C_t)(1-\tau_t^N)z_t F_N(K_t, N_t) - u_\ell(1-N_t) = 0$
  (3) $q_t u_c(C_t) + \beta u_c(C_{t+1}) = 0$
  (4) $C_t + K_{t+1} + G_t = z_t F(K_t, N_t) + (1-\delta)K_t$
  (5) $B_t + G_t = \tau_t^K z_t F_K(K_t, N_t)K_t + \tau_t^N z_t F_N(K_t, N_t)N_t + q_t B_{t+1}$
  for all $t = 0,1,\ldots$, and
$$\lim_{t\to\infty} \beta^t u_c(C_t)K_{t+1} = 0$$
$$\lim_{t\to\infty} \beta^t u_c(C_t)q_t B_{t+1} = 0$$

(4) and (5) imply the agent's budget constraint in equilibrium

(6) $$C_t + K_{t+1} + q_t B_{t+1} = (1-\tau_t^K)z_t F_K(K_t, N_t)K_t + \underbrace{(1-\tau_t^N)z_t F(K_t, N_t)}_{R_t^N} N_t + (1-\delta)K_t + B_t$$

Define
$$R_t^K \equiv (1-\tau_t^K)z_t F_K(K_t, N_t) + 1 - \delta$$
$$R_t^N \equiv (1-\tau_t^N)z_t F_N(K_t, N_t)$$
Then, (6) implies
$$(6)' \quad C_t + K_{t+1} + q_t B_{t+1} = R_t^K K_t + R_t^N N_t + B_t$$

After tax rates of return in equilibrium
$$R_t^K = \frac{u_c(C_{t-1})}{\beta u_c(C_t)}, \qquad \forall t \geq 1$$
$$R_t^N = \frac{u_\ell(1-N_t)}{u_c(C_t)}, \qquad \forall t = 0,1,\ldots$$
$$\frac{1}{q_t} = \frac{u_c(C_t)}{\beta u_c(C_{t+1})}, \qquad \forall t = 0,1,\ldots$$

- Note that R_t^K only holds for $t \geq 1$. This implies that R_0^K is not defined. Since R_0^K is a function of τ_0^K, this means that, depending on how τ_0^K is chosen (which is unconstrained in this case), there could be infinitely many equilibria.
- Since τ_0^K is unconstrained, taxing capital in the initial period is not going to affect agent's marginal decisions. So proportional taxes is the same as a lump-sum tax.
- How much to tax capital in the initial period determines the starting level of the capital of the economy. Government has an incentive to tax a lot in the initial period, because it doesn't have to tax much later.

From $(6)'$,
$$(6)'' \quad C_t + K_{t+1} + \beta\frac{u_c(C_{t+1})}{u_c(C_t)}B_{t+1} = \underbrace{\frac{u_c(C_{t-1})}{\beta u_c(C_t)}}_{[\text{note: } = R_0^K \text{ if } t=0]}K_t + \frac{u_\ell(1-N_t)}{u_c(C_t)}N_t + B_t$$

Multiply $u_c(C_t)$ and sum over all periods on both sides of $(6)''$:

$$\sum_{t=0}^{\infty}\beta^t u_c(C_t)[C_t+K_{t+1}]+\sum_{t=0}^{\infty}\beta^{t+1}u_c(C_{t+1})B_{t+1}$$
$$=\sum_{t=1}^{\infty}\beta^{t-1}u_c(C_{t-1})K_t+u_c(C_0)R_0^K K_0+\sum_{t=0}^{\infty}\beta^t u_\ell(1-N_t)N_t$$
$$+\sum_{t=0}^{\infty}\beta^t u_c(C_t)B_t$$

The debt terms will cancel just like in the previous lectures. So

$$\sum_{t=0}^{\infty}\beta^t u_c(C_t)C_t=u_c(C_0)R_0^K K_0+\sum_{t=0}^{\infty}\beta^t u_\ell(1-N_t)N_t+u_c(C_0)B_0$$
$$\Rightarrow \boxed{\sum_{t=0}^{\infty}\beta^t[u_c(C_t)C_t-u_\ell(1-N_t)N_t]-u_c(C_0)[R_0^K K_0+B_0]=0}$$

This is the ***implementability constraint***, which embeds the equilibrium behavior of the competitive firms and agents in the economy.

- Problem of the government. Given $\{z_t,G_t\}_{t=0}^{\infty}$ and R_0^K, the government solves

$$\max_{\{C_t,K_{t+1},N_t\}_{t=0}^{\infty}}\sum_{t=0}^{\infty}\beta^t u(C_t,1-N_t)$$

subject to

$$C_t+K_{t+1}+G_t=z_tF(K_t,N_t)+(1-\delta)K_t,\qquad \forall t=0,1,\ldots$$
$$\sum_{t=0}^{\infty}\beta^t[u_c(C_t)C_t+u_\ell(1-N_t)N_t]-u_c(C_0)[R_0^K K_0+B_0]=0$$
$$K_0,B_0 \text{ given}$$

Optimal Capital and Labor Taxes (cont'd)

- Problem of the government. Given $\{z_t, G_t\}_{t=0}^{\infty}$, K_0, B_0, and R_0^K, the government solves

$$\max_{\{C_t, K_{t+1}, N_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t, 1 - N_t)$$

subject to

$$C_t + K_{t+1} + G_t = z_t F(K_t, N_t) + (1-\delta)K_t, \qquad \forall t = 0,1, \ldots$$

$$\sum_{t=0}^{\infty} \beta^t [u_c(C_t)C_t - u_\ell(1-N_t)N_t] - u_c(C_0)[R_0^K K_0 + B_0] \underset{(\geq)}{=} 0$$

  - Note that there are not non-negativity constraints, because it is embedded in the implementability constraint when solving the agents' problems.
  - Note it is possible that the second constraint to hold with inequality (in the indicated direction)
  - Recall that

$$R_t^K \equiv (1 - \tau_t^K) z_t F_K(K_t, N_t) + 1 - \delta$$

    By FOC,

$$R_t^K = \frac{u_c(C_{t-1})}{\beta u_c(C_t)}, \qquad \forall t \geq 1$$

    So there is an indeterminacy for R_0^K at $t = 0$. The government has an incentive to tax highly in $t = 0$ to finance future expenditures.

Let $\{\beta^t \mu_t\}_{t=0}^{\infty}$ be the sequence of Lagrange multipliers on the first set of constraints, and λ on the second constraint. The FOC's are

$$C_0: \quad u_c(C_0) - \mu_0 + \lambda\big[u_{cc}(C_0)C_0 + u_c(C_0) - u_{cc}(C_0)[R_0^K K_0 + B_0]\big] = 0$$

$$C_t: \quad \beta^t u_c(C_t) - \beta^t \mu_t + \lambda \beta^t [u_{cc}(C_t)C_t + u_c(C_t)] = 0, \qquad t \geq 1$$

$$K_{t+1}: \quad -\beta^t \mu_t + \beta^{t+1}\mu_{t+1}[z_t F_K(K_{t+1}, N_{t+1}) + 1 - \delta] = 0$$

$$N_t: \quad -\beta^t u_\ell(1-N_t) + \beta^t \mu_t z_t F_N(K_t, N_t) + \lambda \beta^t [u_{\ell\ell}(1-N_t)N_t - u_\ell(1-N_t)] = 0$$

  - *Case 1*. Long-run.
    Suppose $z_t = z^*$ and $G_t = G^*$ for all $t \geq t^* \geq 0$, so that economy converges to a steady state. FOC wrt K_{t+1} becomes

$$-1 + \beta[z^* F_K(K^*, N^*) + 1 - \delta] = 0$$

    Compare this to the agent's Euler equation in and out of steady state:

$$-u_c(C_t) + \beta u_c(C_{t+1})[(1 - \tau_{t+1}^K) z_{t+1} F_K(K_{t+1}, N_{t+1}) + 1 - \delta] = 0$$

$$-1 + \beta[(1 - \tau_t^K) z^* F_K(K^*, N^*) + 1 - \delta] = 0$$

    This means that in the long-run, the government finances the expenditure solely using labor taxes, as it doesn't want to distort the intertemporal trade-off using capital taxes. Note that this holds only in the long-run.

  - *Case 2*. Preference is $u(c, \ell) = \ln c + \alpha \ell$.
    Rewrite the government's problem using this preference:

$$\max_{\{K_{t+1}, N_t\}_{t=0}^{\infty}} \beta^t [\ln[z_t F(K_t, N_t) + (1-\delta)K_t - K_{t+1} - G_t] + \alpha(1 - N_t)]$$

    subject to

$$\sum_{t=0}^{\infty}\beta^t[1-\alpha N_t]-\frac{R_0^K K_0+B_0}{z_0F(K_0,N_0)+(1-\delta)K_0-K_1-G_0}=0$$

FOC's:

$$K_{t+1}:\quad -\frac{\beta^t}{C_t}+\frac{\beta^{t+1}}{C_{t+1}}[z_{t+1}F_K(K_{t+1},N_{t+1})+1-\delta]=0,\qquad t\geq 1$$

$$N_t:\quad \frac{\beta^t}{C_t}z_tF_N(K_t,N_t)-\beta^t\alpha-\beta^t\lambda\alpha=0$$

where

$$C_t=z_tF(K_t,N_t)+(1-\delta)K_t-K_{t+1}-G_t$$

- Note the FOC wrt K_{t+1} indicates that $\tau_t^K=0$ for all $t\geq 2$.
- From the FOC wrt N_t, we derive

$$\frac{z_tF_N(K_t,N_t)}{C_t}-\alpha=\lambda\alpha,\qquad \forall t$$

This implies that τ_t^N is constant over time. Compare this to the agent's FOC:

$$\frac{z_tF_N(K_t,N_t)(1-\tau_t^N)}{C_t}-\alpha=0$$

Incomplete Markets

- Some observations:
  - Agents differ in characteristics: skill, education, employment status, marital status, number of children, etc. → Do any of these matter?
  - Agents face idiosyncratic risk: health, employment, etc.
  - Incomplete markets: agents cannot perfectly insure against idiosyncratic risk.

- Questions / Puzzles (that motivate introduction of incomplete markets and idiosyncratic risk)
  - Income / wealth distribution
  - Consumption and income over the life-cycle
  - Volatility of consumption
  - Wealth of old households
  - Consumption post-retirement
  - Households' asset portfolio
  - Default rates

- A simple economy
  - $T \leq \infty$
  - Storage technology → has a rate of return $r \in \mathbb{R}$ (could be negative), assume $1 + r > 0$
  - No markets
  - Exogenous income

- *Case 1*. Non-stochastic income
  - Problem of the agent:

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t)$$

subject to

$$c_t + a_{t+1} = y_t + a_t(1 + r)$$
$$c_t, a_{t+1} \geq 0$$
$$a_0, \{y_t\}_{t=0}^{T} \text{ given}$$

for all $t = 0,1, \ldots, T$.

  - Some observations of the problem:
    - Could allow for $a_{t+1} \geq a_{MIN,t}$, where $a_{MIN,t}$ could be negative and time-varying (i.e. assets could be negative; but beware of Ponzi-scheme or payback capability)
    - Alternative budget constraint:

$$c_t + q a_{t+1} = y_t + a_t, \qquad q = \frac{1}{1 + r}$$
$$\Leftrightarrow \quad c_t + \frac{a_{t+1}}{1 + r} = y_t + a_t$$

      - This is more like a partial equilibrium version, where there are markets but prices are exogenous.
    - Effects of r. Assume 2 periods, $a_0 = 0$, then we can consolidate the budget constraints in the two periods into a life-time budget constraint:

$$\left.\begin{aligned} c_0 + a_1 &= y_0 \\ c_1 &= y_1 + a_1(1+r) \end{aligned}\right\} \Rightarrow \; c_0 + \frac{c_1}{1+r} = y_0 + \frac{y_1}{1+r}$$

- $\uparrow r$ leads to
  - ♦ (intertemporal) substitution effect $\rightarrow$ c_2 is relatively cheaper
  - ♦ Income effect $\rightarrow$ more disposable income for consumption
  - ♦ "human capital" effect $\rightarrow$ present value income decreases

▪ Focus on interior solution. The FOC is

$$a_{t+1}: \quad -u_c(c_t) + \beta(1+r)u_c(c_{t+1}) = 0, \qquad \forall t = 0,1,\ldots$$

$$\Rightarrow \; u_c(c_{t+1}) = \frac{u_c(c_t)}{\beta(1+r)}$$

$$u_c(c_t) = \frac{u_c(c_{t-1})}{\beta(1+r)} = \frac{u_c(c_{t-2})}{[\beta(1+r)]^2} = \cdots = \frac{u_c(c_0)}{[\beta(1+r)]^t}$$

$$\Rightarrow \; u_c(c_t) = \frac{u_c(c_0)}{[\beta(1+r)]^t}, \qquad \forall t = 0,1,\ldots$$

- $\beta(1+r) = 1 \rightarrow c_t$ is constant over time
- $\beta(1+r) > 1 \rightarrow c_{t+1} > c_t$
- $\beta(1+r) < 1 \rightarrow c_{t+1} < c_t$

Note that
- There is no hump-shape for consumption profile
- r affects the "steepness" of the consumption profile

▪ Present-value wealth

$$t: \quad c_t + a_{t+1} = y_t + a_t(1+r)$$

Start from the last period

$$T: \quad c_T = y_T + a_T(1+r) \qquad \Rightarrow \; a_T = \frac{c_T - y_T}{1+r}$$

$$T-1: \quad c_{T-1} + a_T = y_{T-1} + a_{T-1}(1+r)$$

$$\Rightarrow \; c_{T-1} + \frac{c_T - y_T}{1+r} = y_{T-1} + a_{T-1}(1+r)$$

$$\Rightarrow \; c_{T-1} + \frac{c_T}{1+r} = y_{T-1} + \frac{y_T}{1+r} + a_{T-1}(1+r) \quad \Rightarrow \; a_{T-1}$$

$$T-2: \quad c_{T-2} + \boxed{a_{T-1}} = y_{T-2} + a_{T-2}(1+r)$$

$$\vdots$$

$$\sum_{t=0}^{T} \frac{c_t}{(1+r)^t} = \sum_{t=0}^{T} \frac{y_t}{(1+r)^t} + a_0(1+r)$$

Let the net present value wealth be denoted as

$$w_0 = \sum_{t=0}^{T} \frac{c_t}{(1+r)^t}$$

We have

$$u_c(c_t) = \frac{u_c(c_0)}{[\beta(1+r)]^t}$$
$$\sum_{t=0}^{T} \frac{c_t}{(1+r)^t} = w_0$$

Assume $u(c) = \ln c$. Then,

$$\frac{1}{c_t} = \frac{1}{c_0[\beta(1+r)]^t} \;\Rightarrow\; c_t = c_0[\beta(1+r)]^t$$
$$\sum_{t=0}^{T} \frac{c_0\big(\beta(1+r)\big)^t}{(1+r)^t} = w_0 \;\Rightarrow\; c_0 \sum_{t=0}^{T} \beta^t = w_0$$
$$\Rightarrow\; c_0 \frac{1-\beta^{\mathrm{T}+1}}{1-\beta} = w_0$$
$$\Rightarrow\; c_0 = \gamma(T) w_0, \qquad \gamma(T) = \frac{1-\beta}{1-\beta^{T+1}}$$

$\gamma(T)$ is decreasing in T and $\gamma(T) \to 1-\beta$ as $T \to \infty$.

Suppose $\beta(1+r) = 1$. Then, $c_t = c$ for all t, where

$$c = \gamma(T) w_0$$

Here $\gamma(T)$ is the ***propensity to consume*** out of wealth. If $T = \infty$, then

$$c_t = (1-\beta) w_0, \qquad \forall t$$

This is the ***Permanent Income Hypothesis***.

- *Case 2*. Stochastic Income.
  - Let $s^t = \{y_1, \dots, y_t\}$ denote the history of income realization up to t
  - Let $\pi(s^t)$ be the probability of history s^t with $\sum_{s^t} \pi(s^t) = 1,\ \forall t$.
  - Problem of the agent:

$$\max_{\{c_t(s^t), a_{t+1}(s^t)\}_{\forall t, s^t}} \sum_{t=0}^{T} \sum_{s^t} \beta^t \pi(s^t) u\big(c_t(s^t)\big)$$

subject to

$$c_t(s^t) + a_{t+1}(s^t) = y_t + a_t(s^{t-1})(1+r)$$
$$c_t(s^t), a_{t+1}(s^t) \geq 0$$
$$a_0(s^{-1}), s^{-1} \text{ given}$$

for all t and s^t.

  - Note that budget constraint has to satisfy in every *state of the world* (not on average!).
  - Focus on the interior solution. FOC:

$$-\pi(s^t) u_c\big(c_t(s^t)\big) + \beta(1+r) \sum_{s^{t+1}|s^t} \pi(s^{t+1}) u_c\big(c_{t+1}(s^{t+1})\big) = 0$$
$$\Leftrightarrow\; -u_c\big(c_t(s^t)\big) + \beta(1+r) \sum_{s^{t+1}} \pi(s^{t+1}|s^t) u_c\big(c_{t+1}(s^{t+1})\big) = 0$$
$$\Leftrightarrow\; -u_c(c_t) + \beta(1+r) E_t[u_c(c_{t+1})] = 0$$

The complete markets analog of this condition is

$$-u_c\big(c_t(s^t)\big) + \beta(1+r) u_c\big(c_{t+1}(s^{t+1})\big) = 0, \qquad \forall t, s^t, s^{t+1}$$

The complete markets require that the agent's payoff is a function of the states of the world. So agents can equate intertemporal marginal utilities for every state of the world. However, when the risk is idiosyncratic, the agent can only equate today's marginal utility to the expected marginal utility tomorrow.

Precautionary Savings

- Consider the following example
  - 2 periods, $t \in \{0,1\}$
  - No discounting $\beta = 1$
  - $a_0 = 0$ and $r = 0$
  - y_0 is known, $y_1 = \begin{cases} \bar{y} + \epsilon & \text{with probability } 0.5 \\ \bar{y} - \epsilon & \text{with probability } 0.5 \end{cases}$, with $0 < \epsilon < \bar{y}$.
  - Budget constraints:

$$\left.\begin{aligned} t = 0: \quad & c_0 + a_1 = y_0 \\ t = 1: \quad & c_1 = y_1 + a_1 \end{aligned}\right\} \Rightarrow \; c_0 + c_1 = y_0 + y_1 \;\Rightarrow\; \boxed{c_1 = y_0 + y_1 - c_0}$$

  Let $w \equiv y_0 + \bar{y}$.

  - From FOC of the agent's problem:

$$-u_c(c_t) + \beta(1+r)E_t[u_c(c_{t+1})] = 0 \;\Rightarrow\; u_c(c_0) = E\big(u_c(c_1)\big)$$

  Write everything in terms of c_0:

$$u_c(c_0) = \frac{1}{2}u_c(w + \epsilon - c_0) + \frac{1}{2}u_c(w - \epsilon - c_0)$$

  This equation governs the choice of c_0 (and also savings decision a_0). We are interested in the sign of $\partial c_0 / \partial \epsilon$:

$$u_{cc}(c_0) \cdot \frac{\partial c_0}{\partial \epsilon} = \frac{1}{2}u_{cc}(w + \epsilon + c_0)\left[1 - \frac{\partial c_0}{\partial \epsilon}\right] + \frac{1}{2}u_{cc}(w - \epsilon - c_0)\left[-1 - \frac{\partial c_0}{\partial \epsilon}\right]$$

$$\frac{\partial c_0}{\partial \epsilon}\left[u_{cc}(c_0) + \frac{1}{2}u_{cc}(w + \epsilon - c_0) + \frac{1}{2}u_{cc}(w - \epsilon - c_0)\right]$$

$$= \frac{1}{2}u_{cc}(w + \epsilon - c_0) - \frac{1}{2}u_{cc}(w - \epsilon - c_0)$$

$$\frac{\partial c_0}{\partial \epsilon} = \frac{\frac{1}{2}\big(u_{cc}(w + \epsilon - c_0) - u_{cc}(w - \epsilon - c_0)\big)}{u_{cc}(c_0) + \frac{1}{2}\big(u_{cc}(w + \epsilon - c_0) + u_{cc}(w - \epsilon - c_0)\big)}$$

    - The denominator is negative
    - The numerator is positive if and only if

$$u_{cc}(w + \epsilon - c_0) - u_{cc}(w - \epsilon - c_0) > 0 \;\Leftrightarrow\; u_{ccc} > 0$$

    - Therefore,

$$\frac{\partial c_0}{\partial \epsilon} < 0 \;\Leftrightarrow\; u_{ccc} > 0$$

    If we define $s \equiv w - c_0$, then

$$\frac{\partial s}{\partial \epsilon} > 0 \;\Leftrightarrow\; u_{ccc} > 0$$

    - So people not only saves to smooth consumption, they may also save to insure against the volatility of their future income.

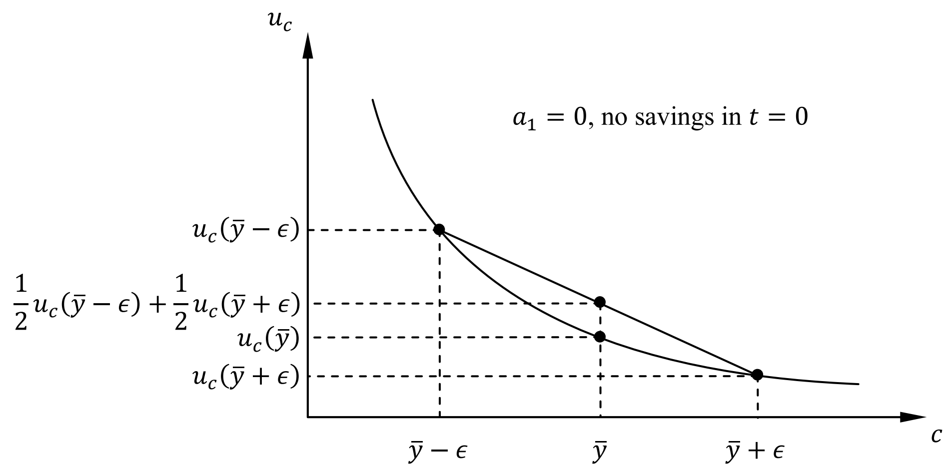
- *Definition.* ***Prudence*** is the propensity to prepare and forearm oneself in the face of uncertainty.
  - Prudence $\rightarrow$ refers to preferences
  - Precautionary savings $\rightarrow$ refers to behavior

- Risk aversion v.s. Prudence
  - Risk aversion refers to u_{cc} (the curvature of the utility function), and requires u to be concave
  - Prudence refers to u_{ccc} (the curvature of the marginal utility), and requires u_c to be convex

- Go back to the FOC

$$u_c(c_0) = E[u_c(c_1)] \;\Rightarrow\; u_c(y_0 - a_1) = \frac{1}{2}u_c(\bar{y} + \epsilon + a_1) + \frac{1}{2}u_c(\bar{y} - \epsilon + a_1)$$

Suppose $y_0 - \bar{y}$.



$$\underbrace{u_c(\bar{y})}_{u_c(c_1)} < \underbrace{\frac{1}{2}u_c(\bar{y} + \epsilon) + \frac{1}{2}u_c(\bar{y} - \epsilon)}_{E[u_c(c_1)] \text{ with no savings}}$$

$$\Rightarrow\; a_1 > 0 \;\Rightarrow \uparrow a_1 \text{ until}$$

$$u_c(\bar{y} - a_1) = \frac{1}{2}u_c(\bar{y} + \epsilon + a_1) + \frac{1}{2}u_c(\bar{y} - \epsilon + a_1)$$

Precautionary Saving induced by Borrowing Constraint

- Problem of the agent

$$\max_{\{c_t(s^t),a_{t+1}(s^t)\}_{t,s^t}} E_0 \sum_{t=0}^{\infty} \beta^t u\big(c_t(s^t)\big)$$

subject to

$$c_t(s^t) + a_{t+1}(s^t) = y_t + a_t(s^{t-1})(1+r)$$
$$c_t(s^t) \geq 0$$
$$a_{t+1}(s^t) \geq 0$$

for all t and all s^t, and $a_0(s^{-1})$, s^{-1} given.

- Note that $s^t = (y_0, \dots, y_t)$.

Assume $c_t(s^t) \geq 0$ does not bind. Let $\lambda_t(s^t)$ be the Lagrange multiplier on the budget constraint, and $\mu_t(s^t)$ on borrowing constraint. The FOC's are

$$c_t(s^t): \qquad \beta^t u_c\big(c_t(s^t)\big) - \lambda_t(s^t) = 0 \quad \forall t, s^t$$
$$a_{t+1}(s^t): \quad -\lambda_t(s^t) + E[\lambda_{t+1}(s^{t+1})|s^t](1+r) + \mu_t(s^t) = 0 \quad \forall t, s^t$$

Combine the two conditions:

$$\lambda_t(s^t) = \beta^t u_c\big(c_t(s^t)\big)$$
$$\Rightarrow \; -\beta^t u_c\big(c_t(s^t)\big) + \beta^{t+1}(1+r)E\big[u_c\big(c_{t+1}(s^{t+1})\big)\big|s^t\big] + \mu_t(s^t) = 0$$
$$\Rightarrow \; u_c\big(c_t(s^t)\big) = \beta(1+r)E\big[u_c\big(c_{t+1}(s^{t+1})\big)\big|s^t\big] - \frac{\mu_t(s^t)}{\beta^t}$$

Using compact notation,

$$u_c(c_t) \geq \beta(1+r)E_t[u_c(c_{t+1})]$$

with equality if and only if the no borrowing constraint $a_{t+1} \geq 0$ does not bind (i.e. $\mu_t = 0$).

$$\left.\begin{array}{c} c_t = y_t + a_t(1+r) - a_{t+1} \\ a_{t+1} \geq 0 \end{array}\right\} \Rightarrow \; c_t \leq y_t + a_t(1+r)$$

$$a_{t+1} = 0 \;\Rightarrow\; c_t = y_t + a_t(1+r) \;\Leftrightarrow\; u_c(c_t) = u_c\big(y_t + a_t(1+r)\big)$$
$$a_{t+1} > 0 \;\Rightarrow\; c_t < y_t + a_t(1+r) \;\Leftrightarrow\; u_c(c_t) > u_c\big(y_t + a_t(1+r)\big)$$

Thus,

$$u_c(c_t) = \max\big\{u_c\big(y_t + a_t(1+r)\big),\; \beta(1+r)E_t[u_c(c_{t+1})]\big\}$$

Assume $\beta(1+r) = 1$, and assume

$$u(c) = -\frac{1}{2}(c-\bar{c})^2 \;\Rightarrow\; \begin{cases} u_c = -(c-\bar{c}) \\ u_{cc} = \quad -1 \\ u_{ccc} = \quad 0 \end{cases}$$

Then,

$$-(c_t - \bar{c}) = \max\{-(y_t + a_t(1+r) - \bar{c}),\; -E_t(c_{t+1} - \bar{c})\}$$
$$\Rightarrow\; -c_t = \max\big\{-\big(y_t + a_t(1+r)\big),\; -E_t(c_{t+1})\big\}$$
$$\Rightarrow\; c_t = \min\{y_t + a_t(1+r),\; E_t[c_{t+1}]\}$$

- Digression. If $u_{ccc}(c) = 0$, then $u_c(c)$ is linear, and so $E[u_c(c)] = u(E[c])$. Thus,

  $$u_c(c_t) = \beta(1+r)E_t[u_c(c_{t+1})] = \beta(1+r)u_c(E[c_{t+1}])$$

  When $\beta(1+r) = 1$,

$$u_c(c_t) = u_c(E_t[c_{t+1}]) \;\Rightarrow\; c_t = E_t[c_{t+1}]$$

Applying this logic, we have

$$c_t = \min\{y_t + a_t(1+r),\; E_t[min\{y_{t+1} + a_{t+1}(1+r),\; E_{t+1}[c_{t+2}]\}]\}$$

- ❖ Results:
  - ➢ If the borrowing constraint never binds,

    $$c_t = E_t[c_{t+j}], \qquad \forall j > 0$$

    That is, c_t follows a Martingale.
  - ➢ Suppose the borrowing constraints bind for $t+j$.

    $$c_{t+j} = y_{t+j} + a_{t+j}(1+r) < E_{t+j}[c_{t+j+1}] \;\Leftrightarrow\; c_{t+j} < E_{t+j}[c_{t+j+1}]$$
    $$\Rightarrow\; c_t < E_t[c_{t+j+1}]$$

    This implies that the agent is going to save in advance to soften the effect of when the borrowing constraint binds $\rightarrow$ this is precautionary savings (even without prudence)!!

- ❖ Implications
  - ➢ Borrowing constraint affect current behavior even if they do not bind today.
  - ➢ If the variance of y_{t+1} increases, then the set of y_{t+1} values for which the borrowing constraint binds also increases

    $$E_t[\min\{y_{t+1} + a_{t+1}(1+r),\; E_{t+1}[c_{t+2}]\}] \text{ decreases}$$

    Hence c_t decreases, savings increase.
    - ▪ If income is more volatile tomorrow (while expected value is the same), then agents potentially face more borrowing constraints in the future. So

      $$\min\{y_{t+1} + a_{t+1}(1+r),\; E_{t+1}[c_{t+2}]\}$$

      is weakly lower, and so

      $$c_t = \min\{y_t + a_t(1+r),\; E_t[\min\{y_{t+1} + a_{t+1}(1+r),\; E_{t+1}[c_{t+2}]\}]\}$$

      is lowered.

Natural Borrowing Limit

- Let $y_t \in [y_L, y_H] \subset (0, \infty)$, and y_t follows a stationary Markov process.
- Let $a_{t+1} \geq \underline{a} \rightarrow$ borrowing constraint
- Budget constraint of the agent:

$$c_t + a_{t+1} = y_t + a_t(1+r)$$

  - Consider an agent with $a_t = \underline{a}$ and that "rolls-it-over", i.e. chooses $a_{t+1} = \underline{a}$. Then,

$$c_t + \underline{a} = y_t + \underline{a}(1+r)$$
$$c_t = y_t + \underline{a}r$$

  - Suppose this agent receives $y_t = y_L$ forever

$$c_t = y_L + \underline{a}r$$
$$c_t \geq 0 \;\;\Rightarrow\;\; y_L + \underline{a}r \geq 0 \;\;\Rightarrow\;\; \underline{a} \geq -\frac{y_L}{r}$$

    This is the lowest possible asset level is

$$\underline{a} = -\frac{y_L}{r}$$

    This is the ***natural borrowing limit***.
    - If the utility function satisfies the Inada conditions, then this borrowing constraint does not bind.

- Recursive problem of the agent
  - $y \in Y = \{y^1, \dots, y^N\}$ where $0 < y^1 < \cdots < y^N$ and $N \geq 2$
  - $\pi(y'|y) = \Pr(y_{t+1} = y' | y_t = y)$
  - $\beta(1+r) < 1$
  - Borrowing constraint: $a' \geq \underline{a}$, where $\underline{a} \geq -y^1/r$
  - Asset state space $[\underline{a}, \bar{a}]$, where $\bar{a}$ is large enough, so that it doesn't bind

$$V(a, y) = \max_{c, a'} u(c) + \beta \sum_{y' \in Y} \pi(y'|y) V(a', y')$$

  subject to

$$c + a' = y + a(1+r)$$
$$c \geq 0$$
$$a' \geq \underline{a}$$

- Suppose shocks are iid, i.e. y follows an iid process

$$\pi(y'|y) = \pi(y')$$

  Consider

$$V(a, y) = \max u(c) + \beta \sum_{y'} \pi(y') V(a', y')$$

  subject to

$$c + a' = y + a(1+r)$$

  But knowing y does not help the agent to predict the future. So can define

$$x \equiv y + a(1+r)$$

  This is the "cash-in-hand", and write the problem as

$$V(x) = \max_{c,a'} u(c) + \beta \sum_{y' \in Y} \pi(y') V \overbrace{\left(y' + a'(1+r)\right)}^{x'}$$

subject to

$$\begin{aligned} c + a' &= x \\ c &\geq 0 \\ a' &\geq \underline{a} \end{aligned}$$

Recursive Competitive Equilibrium (in Incomplete Markets)

- Individual states: a, y
- Aggregate state: $\Phi(a, y)$, a (probability) measure of people that has asset a and income y
  - The two arguments of Φ are both essential because people with the same asset level a but different income levels y are going to make different decisions

  - Let $A = [0, \bar{a}]$ be the set of possible asset level.
  - Let $Y = \{y^1, y^2, \dots, y^N\}$ be the set of possible "income" level
    - This is really the efficiency units of labor or individual productivity.
    - Assume $N \geq 2$ and $0 \leq y^1 \leq \cdots \leq y^N$.
    - Assume $\{y\}$ follows a stationary Markov process, with
$$\pi(y'|y) = \Pr(y_{t+1} = y' | y_t = y)$$
    - Let
$$\begin{aligned} \mathcal{P}_Y &= \text{Power set of } Y \\ \mathcal{B}_A &= \text{set of Borel sets that are subsets of } A \\ S &= A \times Y = \text{set of all outcomes (sample space)} \\ \Sigma_S &= \mathcal{B}_A \times \mathcal{P}_Y = \text{collection of events } (\sigma\text{-algebra}) \end{aligned}$$
      Typical elements of each of these sets are denoted
$$\mathcal{A} \in \mathcal{B}_A, \qquad \mathscr{y} \in \mathcal{P}_Y, \qquad \mathscr{s} \in \Sigma_S$$
      Let
$$\begin{aligned} M &= (S, \Sigma_S) = \text{measurable space} \\ \mathcal{M} &= \text{set of all probability measures over } M \end{aligned}$$
      Φ is an element of $\mathcal{M}$.
      - For any $\mathscr{s} \in \Sigma_S$, $\Phi(\mathscr{s})$ is the measure of agents in the set $\mathscr{s}$.

- Equilibrium functions:
  - Individual functions: $S \times \mathcal{M} \to \mathbb{R}$
  - Aggregate functions (including prices): $\mathcal{M} \to \mathbb{R}$
  - Law of motion for Φ: $\mathcal{M} \to \mathcal{M}$

- Law of motion for Φ $\rightarrow$ need a transition function Q
  - Need to map $\Phi \to \Phi'$
  - Let $Q_\Phi\big((a, y), (\mathcal{A}, \mathscr{y})\big)$ be the probability that an agent with current state (a, y) goes to set $(\mathcal{A}, \mathscr{y})$ next period, given a current distribution Φ.
$$Q_\Phi : \underbrace{S}_{\text{outcome}} \times \underbrace{\Sigma_S}_{\text{events}} \to \underbrace{[0,1]}_{\text{probability}}$$
    The function should look like this
$$Q_\Phi\big((a, y), (\mathcal{A}, \mathscr{y})\big) = \mathbf{1}_{\{a'(a,y,\Phi) \in \mathcal{A}\}} \sum_{y' \in \mathscr{y}} \pi(y'|y)$$
    where $a'(a, y, \Phi)$ is the asset decision rule given Φ.
    - Note that the asset decision rule is <u>endogenous</u>! So the transition equation is also endogenous.

    Then, the law of motion is

$$\Phi'(\mathcal{A},\mathcal{Y}) = \int_S Q_\Phi\big((a,y),(\mathcal{A},\mathcal{Y})\big)d\Phi\,, \qquad \forall(\mathcal{A},\mathcal{Y}) \in \Sigma_S$$
$$= \int_S Q_\Phi\big((a,y),(\mathcal{A},\mathcal{Y})\big)\Phi(da \times dy)$$

In more compact notation, we write

$$\Phi' = H(\Phi)$$

where $H : \mathcal{M} \to \mathcal{M}$.

- Problem of the agent (household)

$$V(a,y,\Phi) = \max_{c,a'} u(c) + \beta \sum_{y' \in Y} \pi(y'|y)V(a',y',\Phi')$$

subject to

$$c + a' = w(\Phi)y + a\big(1 + r(\Phi)\big)$$
$$c, a' \geq 0$$
$$\Phi' = H(\Phi)$$

- Problem of the (representative) Firm

$$\max_{K^f,N^f} F(K^f,N^f) - [r(\Phi) + \delta]K^f - w(\Phi)N^f$$

First-order conditions are

$$F_K(K^f,N^f) - [r(\Phi) + \delta] = 0$$
$$F_N(K^f,N^f) - w(\Phi) = 0$$

  - Market clearing in factor markets

$$K^f = \int_S a\, d\Phi = K(\Phi)$$
$$N^f = \int_S y\, d\Phi = \bar{N}$$

  So in equilibrium, factor prices are

$$r(\Phi) = F_K(K(\Phi),\bar{N}) - \delta$$
$$w(\Phi) = F_N(K(\Phi),\bar{N})$$

- A recursive competitive equilibrium (RCE) is the following list of functions:
  - Value function: $V(a,y,\Phi))$
  - Individual decision rules: $c(a,y,\Phi)$, $a'(a,y,\Phi)$
  - Aggregate policy function: $K(\Phi)$
    - This is optional, just for saving on notations
  - Prices: $r(\Phi)$ and $w(\Phi)$
  - Law of motion for the distribution: $H(\Phi)$

  such that for any $\Phi \in \mathcal{M}$, the following conditions hold:

  (1) Agents maximize utility, i.e. $\forall a \in A,\ \forall y \in Y$,

$$\{c(a,y,\Phi), a'(a,y,\Phi)\} = \arg\max_{c,a'} u(c) + \beta \sum_{y' \in Y} \pi(y'|y)V\big(a',y',H(\Phi)\big)$$

  subject to

$$c + a' = w(\Phi)y + a\big(1 + r(\Phi)\big)$$
$$c, a' \geq 0;$$

and

$$V(a,y,\Phi)=u\big(c(a,y,\Phi)\big)+\beta\sum_{y'\in Y}\pi(y'|y)V\big(a'(a,y,\Phi),y',H(\Phi)\big)$$

(2) Firms maximize profits, i.e.

$$r(\Phi)=F_K(K(\Phi),\bar{N})-\delta$$
$$w(\Phi)=F_N(K(\Phi),\bar{N})$$

(3) Asset and goods markets clear:

$$K\big(H(\Phi)\big)=\int_S a'(a,y,\Phi)\,d\Phi$$

$$\int_S c(a,y,\Phi)\,d\Phi+K\big(H(\Phi)\big)=F(K(\Phi),\bar{N})+(1-\delta)K(\Phi)$$

(4) Law of motion for distribution

$$\Phi'(\mathcal{A},\mathcal{Y})=H(\Phi)(\mathcal{A},\mathcal{Y})=\int_S Q_\Phi\big((a,y),(\mathcal{A},\mathcal{Y})\big)\,d\Phi\,,\qquad \forall(\mathcal{A},\mathcal{Y})\in\Sigma_S$$

where

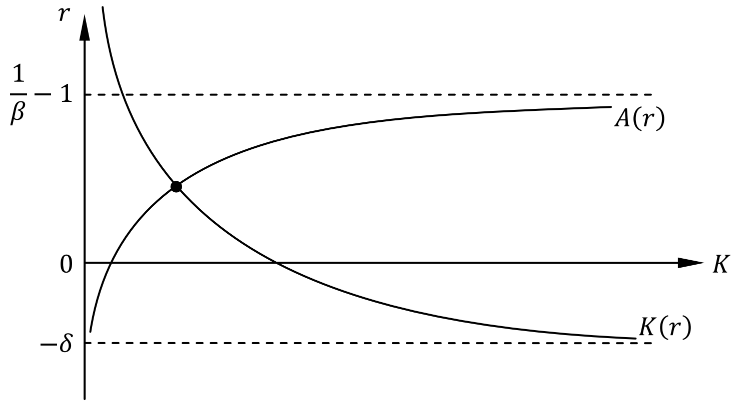
$$Q_\Phi\big((a,y),(\mathcal{A},\mathcal{Y})\big)=\mathbf{1}_{\{a'(a,y,\Phi)\in\mathcal{A}\}}\sum_{y'\in\mathcal{Y}}\pi(y'|y)$$

Stationary RCE

- A SRCE is the following list of functions:
  - Value function: $V(a.y) : S \to \mathbb{R}$
  - Individual decision rules: $c(a,y) : S \to \mathbb{R}$, and $a'(a,y) : S \to \mathbb{R}$
  - Aggregate variable: K
  - Prices: r and w
  - Measure: $\Phi \in \mathcal{M}$

  such that
  1. Agents maximize utility:
  $$\{c(a,y), a'(a,y)\} = \arg\max_{c,a'} u(c) + \beta \sum_{y' \in Y} \pi(y'|y) V(a',y')$$
  subject to
  $$c + a' = wy + a(1+r)$$
  $$c, a' \geq 0$$
  and
  $$v(a,y) = u\big(c(a,y)\big) + \beta \sum_{y' \in Y} \pi(y'|y) V(a'(a,y), y')$$
  for all $a \in A$ and all $y \in Y$.
  2. Firms maximize profits:
  $$r = F_K(K, \bar{N}) - \delta$$
  $$w = F_N(K, \bar{N})$$
  3. Markets clear:
  $$K = \int_S a'(a,y)\, d\Phi$$
  $$F(K, \bar{N}) = \int_S c(a,y)\, d\Phi + \delta K$$
  4. The distribution maps into itself:
  $$\Phi(\mathcal{A}, \mathcal{y}) = \int_S Q\big((a,y), (\mathcal{A}, \mathcal{y})\big)\, d\Phi, \qquad \forall (\mathcal{A}, \mathcal{y}) \in \Sigma_S$$
  where
  $$Q\big((a,y), (\mathcal{A}, \mathcal{y})\big) = \mathbf{1}_{\{a'(a,y) \in \mathcal{A}\}} \sum_{y' \in Y} \pi(y'|y)$$

- Existence and uniqueness of SRCE
  - By Walras Law, we only need to verify that one market clearing condition
    - Objective: show that equilibrium in the asset market exists and is unique, i.e.
    $$\exists!\, r : \underbrace{K(r)}_{\text{demand}} = \underbrace{\int_S a_r'(a,y)\, d\Phi_r}_{\text{supply}}$$
    Where does $K(r)$ come from? From the firm's profit maximization condition:
    $$r = F_K(K(r), \bar{N}) - \delta$$
    Let $A(r) = \int_S a_r'(a,y)\, d\Phi_r$.



- Complete markets: $\beta(1+r)=1$. So in the incomplete market, interest rate has the upper bound $(\beta^{-1}-1)$
- Note that equilibrium interest rate may not be unique, because $A(r)$ is not necessarily monotone [unless substitution effect dominates]
- Continuity of $A(r)$ depends on the existence and uniqueness of Φ_r.